UNIT - II

Fourier Series Analysis :-

Fourier Series representation of Continuous time periodic signals, Trigonometric and Exponential Fourier series, properties of Fourier series.

INTRODUCTION:

All the signals till now were drawn with respect to time (ie) time 't' is a independent variable. The time domain signal is not sufficient for its analysis (ie) we cannot get deep knowledge of a signal in the time domain representation.

Hence, we have to use the frequency domain representation of the signal in order to get deep knowledge about the signal and its characteristics. Thus, analysis of signal is important. Most of the times. It is necessary to convert signal from time domain to frequency domain and vice versa.

For analysing the signal, following parameters are evaluated,

(i) Amplitude
(ii) frequency Content
(iii) power and Energy Content
(iv) periodicity.

* The following Methods used to convert continuous time signals into continuous frequency domain signals are

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(i) FOURIER SERIES
(ii) FOURIER TRANSFORM
(iii) LAPLACE TRANSFORM.

FOURIER SERIES :-

Fourier Series is a mathematical tool to represent a periodic continuous time signal in the form of sum of infinite number of sine and cosine terms.

ie;

Fourier Series representation is used only for periodic continuous signal.

In fourier Series, the periodic CT signals is expanded interms of its harmonics which are sinusoidal and orthogonal to another.

After the fourier Series analysis we obtain the following information about the signal

(i) Their amplitudes.
(ii) Their relative phase difference between those frequency components.

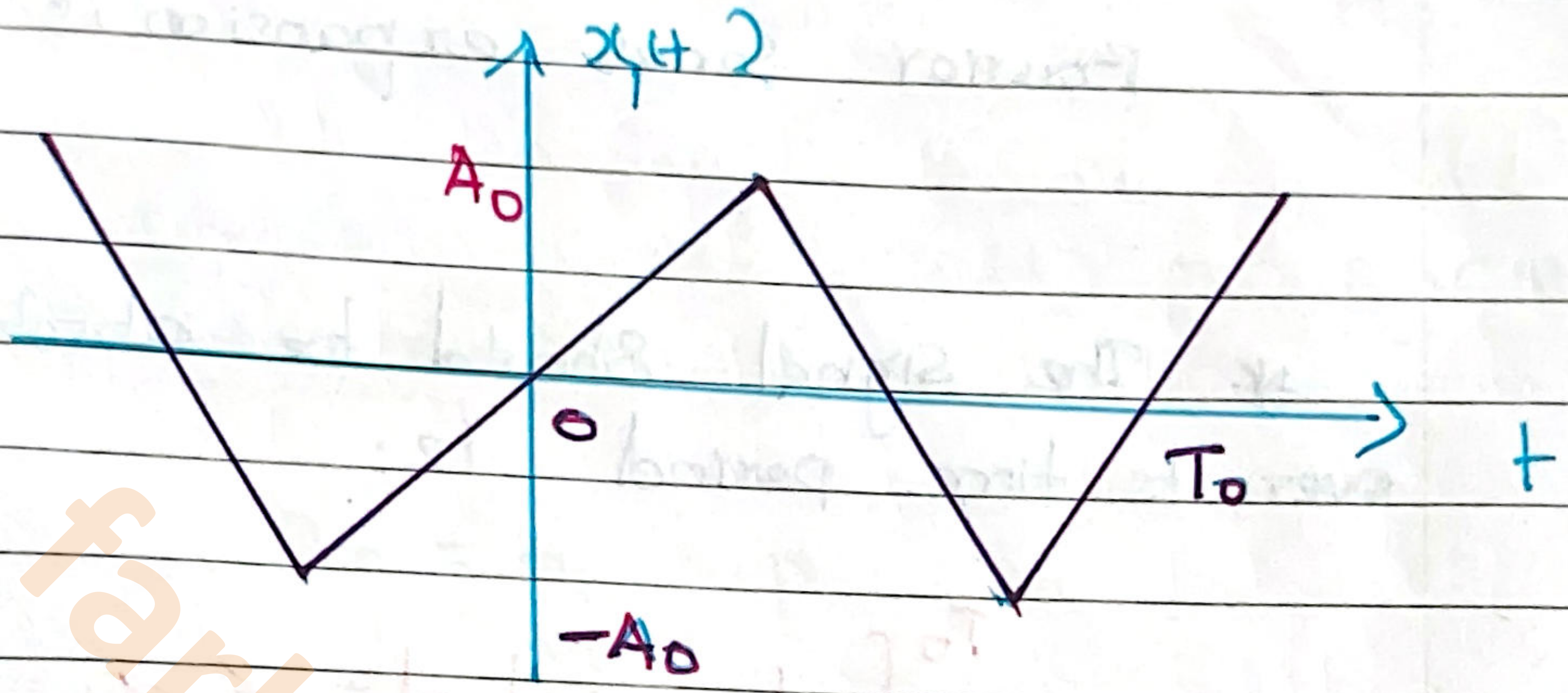
Types of fourier Series :-

There are three types of Fourier Series used for the analysis of periodic signals. They are.

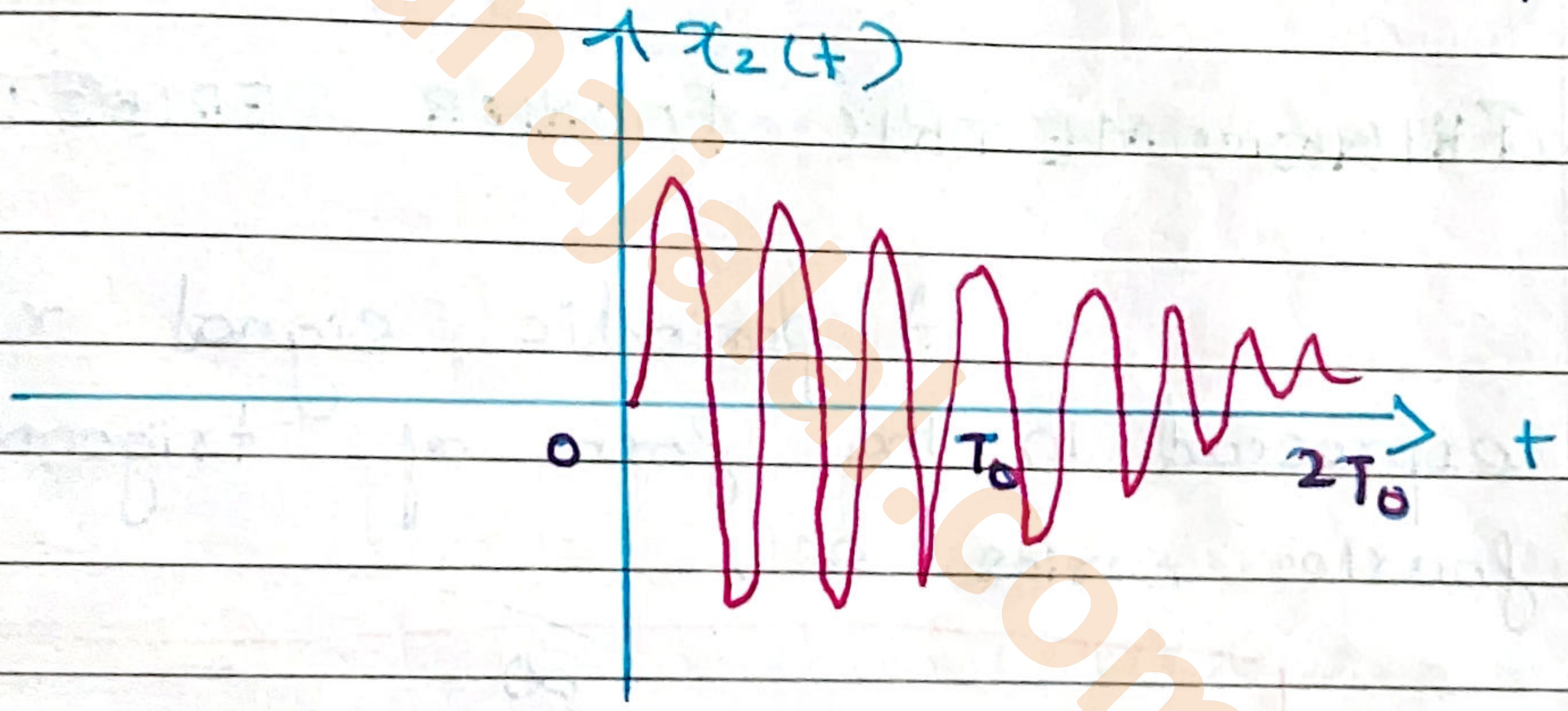
(i) Trigonometric (or) quadrature fourier series.
(ii) polar Fourier Series
(iii) Exponential fourier series.

Condition for exsistance of fourier Series : (DIRICHLET CONDITION).

* Signals should have finite no. of maxima and minima over its time period.

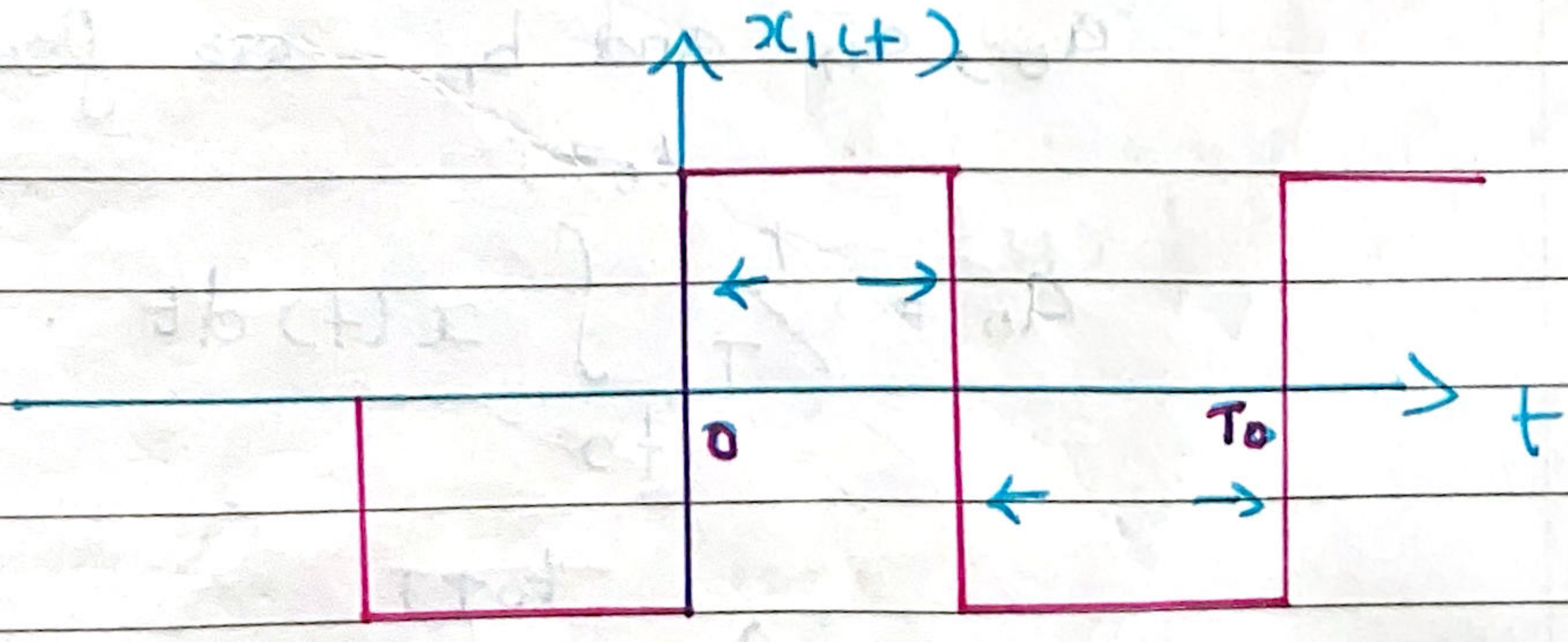


$x_1(t) \rightarrow$ Fourier series expantion is possible.



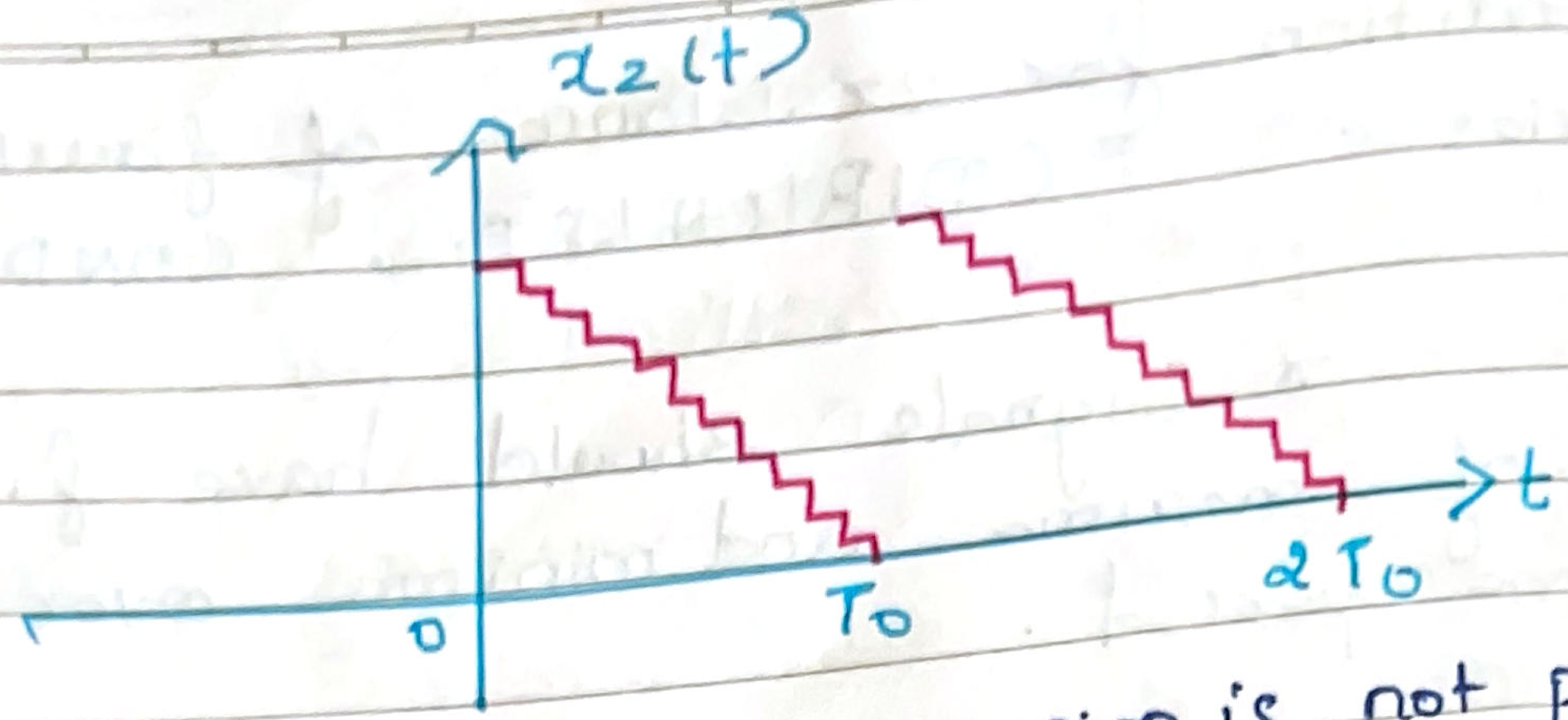
$x_2(t) \rightarrow$ Fourier series expansion is not possible

* The Signal should have finite no. of discontinuties over its time period.



No. of Discontinuties : 2

Fourier Series expansion is possible



Fourier Series expansion is not possible

* The signal should be absolutely integrable over its time period ie:

$$\int_0^{T_0} |x(t)|\, dt < \infty$$

TRIGONOMETRIC FOURIER SERIES :-

A periodic signal $x(t)$ can be expressed in the form of trigonometric fourier series as,

$$\boxed{x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \omega_0 t + \sum_{n=1}^{\infty} b_n \sin \omega_0 t .}$$

Where,

a_0, a_n and b_n are fourier coefficient

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\, dt .$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos n\omega_0 t\, dt$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin n\omega_0 t \, dt$$

where $T = \frac{2\pi}{\omega_0}$ = fundamental time period.

NOTE :-

* If the signal, $x(t)$ is EVEN (i.e) $\boxed{x(-t) = x(t)}$, then

$$a_0 = b_n = 0$$

$$\therefore x(t) = \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

* If the signal, $x(t)$ is ODD i.e;

$\boxed{x(-t) = -[x(t)]}$, then

$$a_n = 0$$

$$\therefore x(t) = a_0 + \sum_{n=1}^{\infty} b_n \sin \omega_0 t.$$

Fourier Series coefficients :-

The constant $a_0, a_1, a_2, \ldots a_n$ & $b_0, b_1, b_2, \ldots b_n$ are called fourier series coefficients :-

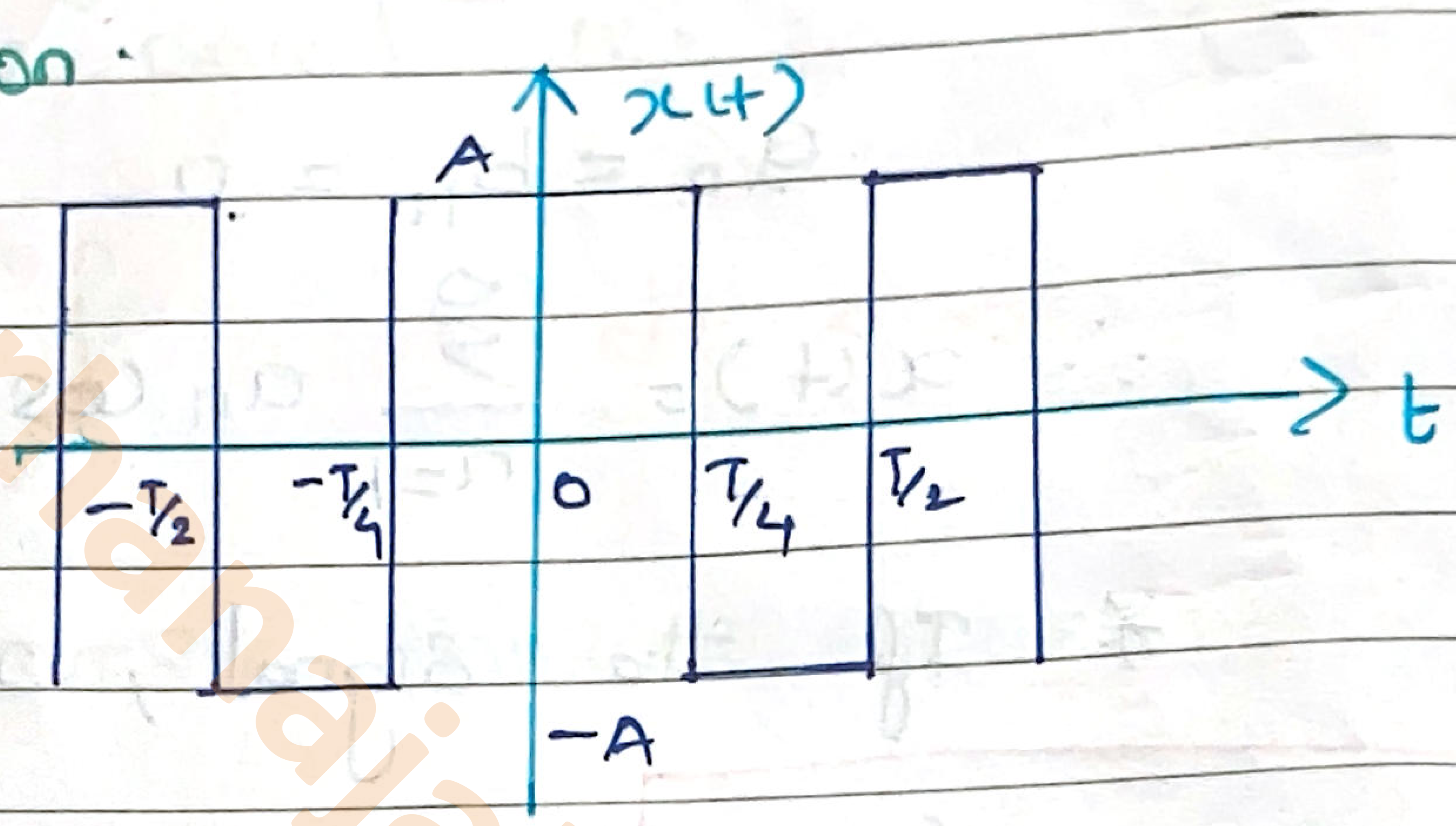
$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) \, dt.$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos(n\Omega_0 t) \, dt$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin(n \Omega_0 t) \, dt$$

Problem:

(1) Fig shows a periodic square wave signal which is symmetrical with respect to the vertical axis. obtain its Fourier series representation.



Solution:-

Given:

$$x(t) = \begin{cases} -A & \text{for } -T/2 < t < -T/4 \\ A & \text{for } -T/4 < t < T/4 \\ -A & \text{for } T/4 < t < T/2 \end{cases}$$

To find:

Fourier series representation of $x(t)$

Given $x(t)$ is an EVEN signal

$\therefore a_0 = b_n = 0$

&

$$x(t) = \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

where

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \cos n\omega_0 t \, dt$$

$$= \frac{2}{T} \left\{ \int_{-T/2}^{-T/4} (-A) \cos n\omega_0 t \, dt + \int_{-T/4}^{T/4} A \cos n\omega_0 t \, dt + \int_{T/4}^{T/2} (-A) \cos n\omega_0 t \, dt \right\}$$

$$a_n = \frac{2A}{T} \left\{ \left[\frac{-\sin n\omega_0 t}{n\omega_0} \right]_{-T/2}^{-T/4} + \left[\frac{\sin n\omega_0 t}{n\omega_0} \right]_{-T/4}^{T/4} + \left[\frac{-\sin n\omega_0 t}{n\omega_0} \right]_{T/4}^{T/2} \right\}$$

w.k.t $\int \cos(at)\,dt = \frac{1}{a} \sin(at)$

$$a_n = \frac{2A}{n\omega_0 T} \left\{ -\sin\left(\frac{-n\omega_0 T}{4}\right) + \sin\left(\frac{-n\omega_0 T}{2}\right) + \sin\left(\frac{n\omega_0 T}{4}\right) - \sin\left(\frac{-n\omega_0 T}{4}\right) - \sin\left(\frac{n\omega_0 T}{2}\right) + \sin\left(\frac{n\omega_0 T}{4}\right) \right\}$$

$$a_n = \frac{8A}{n\omega_0 T} \sin\left(\frac{n\omega_0 T}{4}\right) - \frac{4A}{n\omega_0 T} \sin\left(\frac{n\omega_0 T}{2}\right)$$

W.K.T $T = \frac{2\pi}{\omega_0}$

$$a_n = \frac{8A}{n\omega_0 \times \frac{2\pi}{\omega_0}} \sin\left(\frac{n\omega_0 \times \frac{2\pi}{\omega_0}}{4}\right) -$$

$$\frac{4\cancel{8}\pi}{n\omega_0 \frac{2\pi}{\omega_0}} \sin\left(\frac{n\omega_0 \times \frac{2\pi}{\omega_0}}{2}\right)$$

$$a_n = \frac{4A}{n\pi} \sin\left(\frac{n\pi}{2}\right) - \frac{2A}{n\pi} \sin(n\pi)$$

$\because \sin n\pi = 0$

$$a_n = \frac{4A}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

$$x(t) = \sum_{n=1}^{\infty} \frac{4A}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos \omega_0 t$$

2) Find the Fourier Series representation of given signal $x(t) = t$; $0 \le t \le 2$.

Given :-

$x(t) = t$; $0 \le t \le 2$

To find :

Fourier Series rep.

Solution:

W.K.T

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \omega_0 n t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

To find a_0 :

W.K.T $$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\, dt$$

$$= \frac{1}{2}\int_0^2 t\, dt \qquad \because T=2$$

$$= \frac{1}{2}\left[\frac{t^2}{2}\right]_0^2$$

$$= \frac{1}{2}\left[\frac{4}{2}\right] = 1$$

$$\boxed{a_0 = 1}$$

To find a_n :

w.k.T $\quad a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos n\omega_0 t\, dt$

$$= \frac{2}{2}\int_0^2 t \cos n\omega_0 t\, dt$$

$\because \int uv\,dx = uv_1 - u'v_2 + u''v_3 - \dots$

Let $u = t$; $v = \cos n\omega_0 t\, dt$

$u' = 1$; $v_1 = \frac{\sin n\omega_0 t}{n\omega_0}$

$u'' = 0$;

$v_2 = \frac{-\cos n\omega_0 t}{(n\omega_0)^2}$

$$a_n = \left[\frac{t \sin n\omega_0 t}{n\omega_0} + \frac{\cos n\omega_0 t}{(n\omega_0)^2}\right]_0^2$$

$$= \left[\frac{2\sin n\omega_0(2)}{n\omega_0} + \frac{\cos n\omega_0(2)}{(n\omega_0)^2} - \frac{1}{(n\omega_0)^2}\right]$$

$\because \omega_0 = 2\pi T \implies \omega_0 = 4\pi$

$$= \frac{2\sin 4\pi n(2)}{n\omega_0 4\pi} + \frac{\cos n4\pi(2)}{(n\underset{4\pi}{\omega_0})^2} - \frac{1}{(n\underset{4\pi}{\omega_0})^2}$$

W.K.T $\sin 4\pi = 0$ & $\cos 4\pi = 1$

$$= 0 + \frac{1}{(4n\pi)^2} - \frac{1}{(4n\pi)^2}$$

$$\boxed{a_n = 0}$$

To find b_n

W.K.T $b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin n\omega_0 t\, dt$

$$= \frac{2}{2}\int_0^2 t\sin\omega_0 t\, dt.$$

$$= \int_0^2 t\sin n\omega_0 t\, dt.$$

$u = t$ $\quad v = \sin n\omega_0 t\, dt$

$u' = 1$ $\quad v_1 = \frac{-\cos n\omega_0 t}{n\omega_0}$

$u'' = 0$ $\quad v_2 = \frac{-\sin n\omega_0 t}{(n\omega_0)^2}$

$$b_n = \left[\frac{-t\cos n\omega_0 t}{n\omega_0} + \frac{\sin n\omega_0 t}{(n\omega_0)^2}\right]_0^2$$

$$= \frac{-2\cos n\omega_0(2)}{n\omega_0} + \frac{\sin n\omega_0(2)}{(n\omega_0)^2}$$

$\omega_0 = 4\pi$

$$b_n = \frac{-2\cos 8\pi n}{n(4\pi)} + \frac{\cancel{\sin 8\pi n}^{\,0}}{(n4\pi)^2}$$

$$\boxed{b_n = \frac{-2}{4\pi n}}$$

$$x(t) = 1 + \sum_{n=1}^{\infty} \frac{-2}{4\pi n} \sin n\omega_0 t .$$

Exponontial Fourier Series

Any physically realizable "periodic signal" with period T, can be represented as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T}$$

Multiply eqn by $e^{-jk\omega_0 t}$ & integrate over one period.

$$\int_{t_0}^{t_0+T} x(t)\, e^{-jk\omega_0 t}\, dt = \int_{t_0}^{t_0+T} \left[\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}\right] e^{-jk\omega_0 t}\, dt .$$

$$= \sum_{n=-\infty}^{\infty} c_n \int_{t_0}^{t_0+T} e^{jn\omega_0 t}\, e^{-jk\omega_0 t}\, dt$$

Sub the relation

$$\int_{t_0}^{t_0+T} e^{jn\omega_0 t} e^{-jk\omega_0 t} dt = \begin{cases} 0 & k \neq n \\ T & k = n \end{cases}$$

$$\int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt = T c_K .$$

(or)

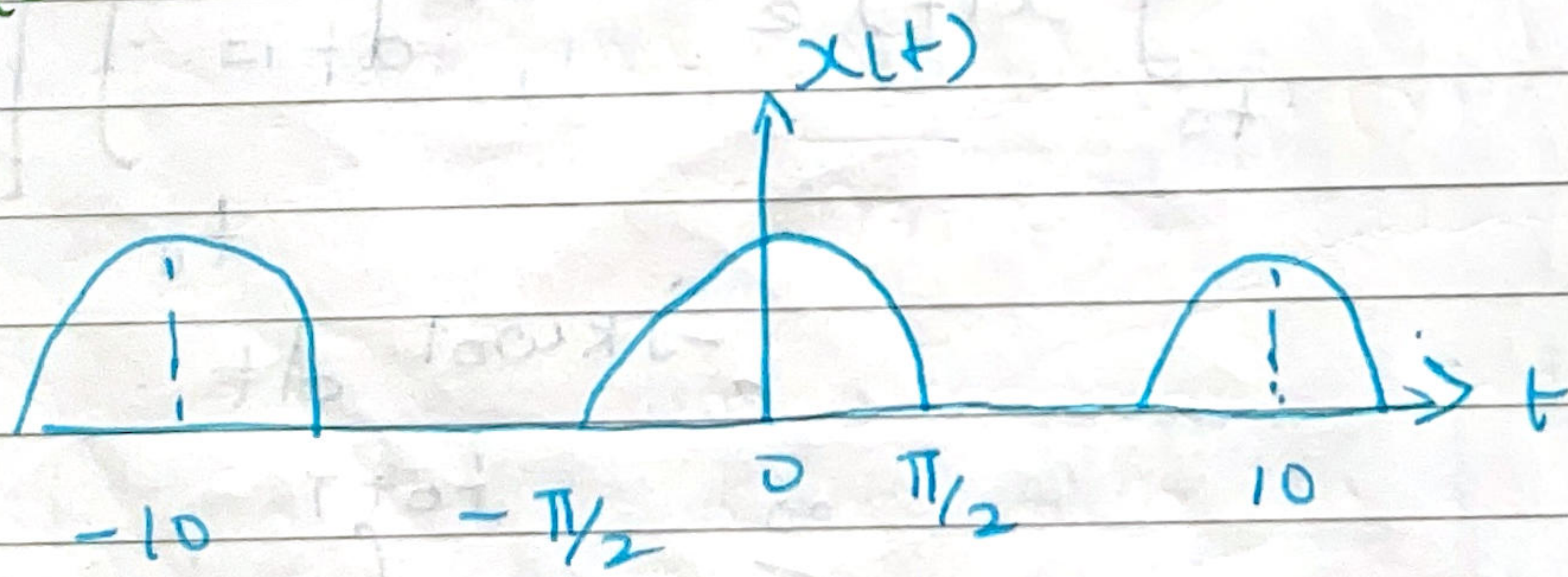
$$c_K = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt$$

$$\boxed{c_n = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jn\omega_0 t} dt}$$

where c_n are the fourier series coefficients of exponential fourier series.

Problems :-

1) Find the exponential series of the following signal.



Given :-

$T = 10$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{10} = \pi/5$$

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$$x(t) = \cos t \text{ for } -\pi/2 < t < \pi/2$$

$$c_0 = 1/T \int_{t_0}^{t+T} x(t)\,dt$$

$$= 1/10 \int_{-\pi/2}^{\pi/2} \cos t\,dt = 1/10 \left[\sin t\right]_{-\pi/2}^{\pi/2}$$

$$= 2/10 = 1/5$$

$$\boxed{c_0 = 1/5}$$

$$c_n = 1/10 \int_{-\pi/2}^{\pi/2} \cos t\, e^{-jn\omega_0 t}\,dt$$

w.k.T

$$\boxed{\cos t = \frac{e^{jt} + e^{-jt}}{2}}$$

$$c_n = 1/10 \int_{-\pi/2}^{\pi/2} \left(e^{jt(1-n\Omega_0)} + e^{-jt(1+n\Omega_0)}\right) dt$$

$$= 1/20 \frac{1}{j(1-n\Omega_0)} \left\{e^{j\pi/2(1-n\Omega_0)} - e^{-j\pi/2(1-n\Omega_0)}\right.$$

$$+ \frac{1}{-j(1+n\Omega_0)} \left\{e^{-j\pi/2(1+n\omega_0)} - e^{j\pi/2(1+n\omega_0)}\right.$$

$$= 1/10 \left[\frac{1}{1-n\omega_0} \sin \pi/2 (1-n\omega_0) + \right.$$

$$\frac{1}{1+n\omega_0} \sin \pi/2 (1+n\omega_0)$$

$$= \frac{1}{10(1-n^2\omega_0^2)}\Big[(1+n\omega_0)\left(\cos \tfrac{\pi}{2} n\omega_0\right)$$

$$+ (1-n\omega_0^{\cancel{2}}) \cos \tfrac{\pi}{2}\left(\tfrac{\pi}{2} n\omega_0\right)\Big]$$

$$= \frac{1}{10(1-n^2\omega_0^2)}\left(2\cos \tfrac{\pi}{2} n\omega_0\right)$$

$$= \frac{1}{5(1-n^2\omega_0^2)} \cos\left(\tfrac{\pi}{2} n\omega_0\right)$$

For $\cancel{\omega}$ $\omega_0 = \pi/5$

$$C_n = \frac{1}{5\left(1-n^2(\pi/5)^2\right)} \cos\left\{\tfrac{\pi}{2} n\left(\tfrac{\pi}{5}\right)\right\}$$

$$\boxed{C_n = \frac{\cos \frac{\pi^2 n}{10}}{5\left[1-\left(\frac{n\pi}{5}\right)^2\right]}}$$

2) The complex exponential representation of a signal $x(t)$ over the time interval $(0,T)$ is $x(t) = \sum_{n=-\infty}^{\infty} \left(\frac{3}{4} + (n\pi)^2\right) e^{jn\pi t}$

i) What is the numerical value of T ?
ii) One of the components of $x(t)$ is $A\cos 3\pi t$. Determine the value of A.

Solution :-

Given:-

(i) $$x(t) = \sum_{n=-\infty}^{\infty} \left(\frac{3}{4} + (n\pi)^2\right) e^{jn\pi t}$$

Compare with eqn

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{jn\omega_0 t}$$

$$\omega_0 = \pi$$

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\pi} = 2$$

$$\boxed{T = 2}$$

(ii) $$x(t) = \sum_{n=-\infty}^{\infty} \left(\frac{3}{4} + (n\pi)^2\right) e^{jn\pi t}$$

The coefficients of $\cos 3\pi t$ can be obtained by expanding the above series.

$$x(t) = \dots + \left(\frac{3}{4} + (-3\pi)^2 e^{-j3\pi t}\right) + \left(\frac{3}{4} + (-2\pi)^2\right) e^{-j2\pi t} + \dots + \left(\frac{3}{4} + (2\pi)^2\right) e^{j2\pi t} + \left(\frac{3}{4} + (3\pi)^2\right) e^{j3\pi t} + \dots$$

$$= \dots 2\left(\frac{3}{4} + 9\pi^2\right) \cos 3\pi t + 2\left(\frac{3}{4} + 4\pi^2\right) \cos 2\pi t + \dots$$

The coefficient of $\cos 3\pi t$ is

$$2\left(\frac{3}{4} + 9\pi^2\right)$$

Properties of Fourier Series :-

If $x(t)$ is a periodic signal with period T and fundamental frequency $\omega_0 = 2\pi/T$ then the fourier coefficients of $x(t)$ are denoted by C_n.

The fourier coefficients.

$$FS[x(t)] = C_n$$

$$x(t) \xleftrightarrow{FS} C_n$$

1) Linearity :-

Consider two signal $x(t)$ and $y(t)$ with period T, let the linear combination of these two signals be

$$A x_1(t) + B x_2(t)$$

The linear combination of two signals with period T, the fourier coefficients are C_n and d_n for $x(t)$ & $y(t)$ respectively.

The Fourier ~~series of~~ coefficients of their linear combination are.

$$FS[A x_1(t) + B x_2(t)] = A c_n + B d_n$$

Time shifting :-

In Fourier Series coefficients of $x(t)$ are C_n; the fourier series coefficients of the shifted signal $x(t-t_0)$ are.

$$FS[x(t-t_0)] = e^{-jn\omega_0 t_0} C_n$$

Proof :-

$$FS[x(t-t_0)] = \frac{1}{T}\int_T x(t-t_0)e^{-jn\omega_0 t}\,dt$$

Let $t - t_0 = p$, then

$$FS[x(t-t_0)] = \frac{1}{T}\int x(p)\, e^{-jn\omega_0 t_0}\, e^{-jn\omega_0 p}\, dp$$

$$= e^{-jn\omega_0 t_0}\, \frac{1}{T}\int_T x(p)\, e^{jn\omega_0 p}$$

$$= e^{-jn\omega_0 t_0}\, C_n$$

$$FS[x(t-t_0)] = e^{-jn\omega_0 t_0}\, C_n.$$

Time shift of the signal does not changes the magnitude of the but change their phases.

Time Reversal :-

The time reversal of the signal $x(t)$ is given by

$$FS[x(-t)] = C_{-n}$$

Proof :-

let $y(t) = x(-t)$

$$x(-t) = \sum_{n=-\infty}^{\infty} C_n e^{-jn\omega_0 t}$$

$n = -m$;

$$x(-t) = \sum_{m=-\infty}^{\infty} C_{-m} e^{jm\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} C_{-n} e^{jn\omega_0 t}.$$

$\therefore FS[x(-t)] = C_{-n}$

$x(t)$ even; the sequence of coefficients is also even
$x(t)$ odd; the sequence of coefficients is also odd

Time Scaling:- The time scaling signal of $x(t)$ is denoted as $x(at)$.

* If $a<1$; the resulting time scaled signal is expanded version of $x(t)$.
* If $a>1$; the resulting signal $x(at)$ is compressed version of $x(at)$

$$FTp \text{ of } x(t) = T$$
$$FTp \text{ of } x(at) = T/a$$

$$FS[x(at)] = C_n$$

Convolution:
If $x_1(t)$ and $x_2(t)$ are both periodic with T. The fourier series coefficients is:

$$FS[x_1(t) * x_2(t)] = T\, C_n d_n$$

Conjugation:-
If the fourier coefficients of $x(t)$ are C_n, then the time reversal complex conjugate Fourier series coefficients is.

$$FS[x^*(t)] = C^*_{-n}$$

Proof:-

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jw_0 nt}$$

Applying Conjugation on both sides

$$x^*(t) = \sum_{n=-\infty}^{\infty} c_n^* e^{-j\omega_0 nt}.$$

$$= \sum_{n=-\infty}^{\infty} c_{-n}^* e^{j\omega_0 nt}$$

$$FS[x^*(t)] = c^*_{-n}$$

* If $x(t)$ is ~~real~~ ~~and~~ even, then
$x(t) = x(-t) \Rightarrow c_n = c_{-n}$

* If $x(t)$ is real then, $c_n = c^*_{-n}$
$\Rightarrow c_n = c^*_n \quad \therefore c_n$ is real for all n.

$\Rightarrow$ If $x(t)$ is real & even c_n also real & even.

* If $x(t)$ is odd, $x(t) = -x(-t) \Rightarrow$
$c_n = -c_{-n}$

* If $x(t)$ is real $c_n = c^*_{-n} \Rightarrow$
$c_n = -c^*_n \quad \therefore c_n$ is imaginary for all n.

$\Rightarrow$ If $x(t)$ is real & odd, c_n is purely imaginary & odd.