UNIT - III

Fourier Transform & Laplace Transform :-

Fourier transform of aperiodic signals, Standard signals and periodic signals, properties of fourier transforms. Hilbert transform and its properties. Laplace transforms, ROC, properties, Inverse Laplace transform.

Fourier Transform :-

Fourier transform is a mathematical tool for converting continuous time domain signal into continuous frequency.

The fourier transform of any CT signal, $x(t)$ can be represented as

$$F[(x(t))] = X(j\omega)$$

and defined as,

$$F\{x(t)\} = X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \quad \text{--- (1)}$$

(or)

$$F\{x(t)\} = X(j\omega) = \int_{0}^{\infty} x(t)\, e^{-j\omega t}\, dt \quad \text{--- (2)}$$

* Equation (1) is also called as "two sided Fourier transform" (or) "BILATERAL Fourier transform".

* Equation (2) is also called as "ONE SIDED FOURIER TRANSFORM" (or) "UNILATERAL FOURIER TRANSFORM".

Page No.:
Date:

Existance of Fourier Transform:-

* The CT signal, $x(t)$ should have finite no. of maxima & minima within a period.
* The CT signal, $x(t)$ should have finite no. of discontinuties within a period.
* The CT signal, $x(t)$ should be a integrable ie;

$$\int_{-\infty}^{\infty} x(t)\,dt < \infty$$

INVERSE FOURIER TRANSFORM:-

The inverse fourier transform is used to convert frequency domain ($j\omega$ domain) signal, $X(j\omega)$ to the time domain signal $x(t)$.

It can be represented as

$$F^{-1}[X(j\omega)] = x(t)$$

and defined as,

$$x(t) = F^{-1}[X(j\omega)] = \frac{1}{2\pi j}\int_{-\pi}^{\pi} X(j\omega)\, e^{j\omega n}\, d\omega$$

Problems:-

1] Find the Fourier transform of the following Standard signals.

i) $u(t)$ ii) $\delta(t)$ iii) $r(t)$

Solution:-

i) Given:

$$u(t) = \begin{cases} 1; & t \geq 0 \\ 0; & t < 0 \end{cases}$$

To find:

$$FT[u(t)] = U(j\omega) = ?$$

W.K.T

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\Rightarrow FT[u(t)] = \int_{-\infty}^{\infty} u(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} 1 \times e^{-j\omega t}\, dt$$

$e^{\infty} = \infty$

$e^{-\infty} = 0$

$e^{0} = 1$

$$= \left[\frac{e^{-j\omega t}}{-j\omega}\right]_0^{\infty}$$

(io)

$$\boxed{FT[u(t)] = \frac{1}{j\omega}}$$

ii) Given :-

$\delta(t) = 1 \; ; \; t = 0$

$\quad\quad 0 \; ; \; t \neq 0$

To find :

$$FT[\delta(t)] = \Delta(j\omega) = ?$$

Solution :

$$FT[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= \int \delta(t)\, e^{-j\omega t}\, dt \Big/_{t=0}$$

$$\boxed{FT[\delta(t)] = 1}$$

iii) Given:

$r(t) = t \; ; \; t \geq 0$

$\quad\quad 0 \; ; \; t < 0$

To find :

$$FT[r(t)] = R[j\omega] = ?$$

Page No.:
Date:

Solution :-

$$F[r(t)] = \int_{-\infty}^{\infty} r(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} t\, e^{-j\omega t}\, dt$$

$$\int uv\,dx = uv_1 - u'v_2$$

Let

$u = t \qquad dv = e^{-j\omega t}\, dt$

$u' = 1 \qquad v_1 = \frac{e^{-j\omega t}}{-j\omega}$

$u'' = 0$

$v_2 = \frac{e^{-j\omega t}}{(j\omega)^2}$

$$F[r(t)] = \left[\frac{t\, e^{-j\omega t}}{-j\omega}\right]_0^{\infty} - \left[\frac{e^{-j\omega t}}{(j\omega)^2}\right]_0^{\infty}$$

$$\boxed{F[r(t)] = \frac{1}{(j\omega)^2}}$$

PROPERTIES OF FOURIER TRANSFORM :

1] Linearity property :-

Statement :

If $FT[x_1(t)] = X_1(j\omega)$ &
$FT[x_2(t)] = X_2(j\omega)$

then $FT[a_1 x_1(t) + a_2 x_2(t)] =$
$a_1 X_1(j\omega) + a_2 X_2(j\omega)$

Proof :-

w.r.t $F\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

$$\Rightarrow FT\{a_1 x_1(t) + a_2 x_2(t)\} = \int_{-\infty}^{\infty} [a_1 x_1(t) + a_2 x_2(t)]\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} a_1 x_1(t)\, e^{-j\omega t}\, dt + \int_{-\infty}^{\infty} a_2 x_2(t) e^{-j\omega t}\, dt$$

$$= a_1 \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t}\, dt + a_2 \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t}\, dt$$

$$= a_1\, FT[x_1(t)] + a_2\, FT[x_2(t)]$$

$$\boxed{FT[a_1 x_1(t) + a_2 x_2(t)] = a_1 X_1(j\omega) + a_2 X_2(j\omega)}$$

2] TIME Shifting Property :-

Stmt :-

If $FT[x(t)] = X(j\omega)$

then

$$FT[x(t - t_0)] = e^{-j\omega t_0} X(j\omega)$$

Proof :-

W.K.T $FT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$.

$$\Rightarrow FT = [x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j\omega t}\, dt. \quad \text{— (1)}$$

let

$t - t_0 = \tau \Rightarrow t = \tau + t_0$

$dt = d\tau$

If $t = -\infty$ then $\tau = -\infty$

$t = \infty$ then $\tau = \infty$

$$(1) \Rightarrow FT[x(t-t_0)] = \int_{\tau=-\infty}^{\infty} x(\tau)\, e^{-j\omega(\tau+t_0)}\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} e^{-j\omega t_0}\, d\tau$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau$$

Changing dummy variable 'τ' into original variable 't'

$$\therefore FT[x(t-t_0)] = e^{-j\omega t_0} \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

$$= e^{-j\omega t_0}\, FT[x(t)]$$

$$\boxed{FT[x(t-t_0)] = e^{-j\omega t_0} X(j\omega)}$$

3. Time Scaling property :-

Stmt :-

If $FT[x(t)] = X(j\omega)$
then $FT[x(at)] = \frac{1}{a} X(j\omega/a)$

Proof :-

W.K.T

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt.$$

$$\Rightarrow FT[x(at)] = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt \quad \text{—— (1)}$$

let

$$at = \tau \Rightarrow t = \tau/a$$

$a\,dt = d\tau \quad \Rightarrow \quad dt = d\tau/a$

If $t = -\infty$ then $\tau = -\infty$
$t = \infty$ then $\tau = \infty$

$$(1) \Rightarrow FT[x(at)] = \int_{\tau=-\infty}^{\infty} x(\tau)\, e^{-j\omega(\tau/a)}\, d\tau/a$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)\, e^{-(j\omega/a)\tau}\, d\tau$$

Changing 'τ' into 't'

$$\therefore FT[x(at)] = \frac{1}{a}\int_{-\infty}^{\infty} x(t)\, e^{-(j\omega/a)t}\, dt$$

$$\boxed{FT[x(at)] = \frac{1}{a} X(j\omega/a)}$$

A] FREQUENCY SHIFTING PROPERTY:

stmt:

If $FT[x(t)] = X(j\omega)$

then

$$FT[e^{j\omega_0 t} x(t)] = X(j\omega - j\omega_0)$$

Proof:

w.k.t $FT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

$$\Rightarrow FT[x(t)\, e^{j\omega_0 t}] = \int_{-\infty}^{\infty} x(t)\, e^{j\omega_0 t}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\omega_0)t}\, dt$$

$$\boxed{FT[e^{j\omega_0 t} x(t)] = X[j(\omega-\omega_0)]}$$

Page No.: Date: YOUVA

TIME DIFFERENTIATION PROPERTY :

Stmt :

If $F[x(t)] = X(j\omega)$

then

$$FT\left[\frac{d\,x(t)}{dt}\right] = j\omega\, X(j\omega)$$

Proof :-

w.k.T

$$FT[x(t)] = \int_{0}^{\infty} x(t) e^{-j\omega t}\, dt.$$

$$\Rightarrow FT\left[\frac{d}{dt}x(t)\right] = \int_0^\infty \frac{d}{dt}x(t)\, e^{-j\omega t}\, dt$$

$$= \int_0^\infty e^{-j\omega t}\, d\,x(t) \quad \text{--- (1)}$$

w.k.T

$$\int u\,dv = uv - \int v\,du$$

let

$u = e^{-j\omega t}$ $\qquad dv = d\,x(t)$

$du = e^{-j\omega t}(-j\omega)dt$ $\qquad v = \int d\,x(t) = x(t)$

$$(1) \Rightarrow FT\left[\frac{d}{dt}x(t)\right] = \left[x(t)\,e^{-j\omega t}\right]_0^\infty - \int_0^\infty x(t)\, e^{-j\omega t}(-j\omega)\,dt$$

$$= j\omega \int_0^\infty x(t)\, e^{-j\omega t}\, dt$$

$$= j\omega\; FT[x(t)]$$

$$\boxed{FT\left[\frac{d}{dt}x(t)\right] = j\omega\, X(j\omega)}$$

FREQUENCY DIFFERENTIATION PROPERTY:

Stmt:

If $FT[x(t)] = X(j\omega)$

then

$$FT[t\,x(t)] = -\frac{d}{d(j\omega)} X(j\omega)$$

Proof:-

w.k.T $FT[x(t)] = X(j\omega)$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Differentiate on bothsides w.r.to $(j\omega)$

$$\therefore \frac{d}{d(j\omega)} X(j\omega) = \frac{d}{d(j\omega)} \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t) \frac{\partial}{\partial(j\omega)} e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t) -t\, e^{-j\omega t}\, dt$$

$$= -\int_{-\infty}^{\infty} t\, x(t)\, e^{-j\omega t}\, dt$$

$$\Rightarrow -\frac{d}{d(j\omega)} X(j\omega) = \int_{-\infty}^{\infty} t\, x(t)\, e^{-j\omega t}\, dt$$

$$\therefore FT[t\,x(t)] = -\frac{d}{d(j\omega)} X(j\omega)$$

TIME REVERSAL PROPERTY:

Stmt:

If $FT[x(t)] = X(j\omega)$

then $FT[x(-t)] = X(-j\omega)$

Proof:-

w.k.T $\quad FT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

$$\Rightarrow FT[x(-t)] = \int_{-\infty}^{\infty} x(-t)\, e^{-j\omega t}\, dt \quad \text{——— } (1)$$

Let

$$-t = \tau \quad \Rightarrow t = -\tau$$

$$-dt = d\tau \quad \Rightarrow dt = -d\tau$$

If $t = \infty$ then $\tau = -\infty$

$t = -\infty$ then $\tau = \infty$

$$(1) \Rightarrow FT[x(-t)] = \int_{\tau=-\infty}^{\infty} x(\tau)\, e^{-j\omega(-\tau)}\, (-d\tau)$$

$$= -\int_{\tau=-\infty}^{\infty} x(\tau)\, e^{-j\omega(-\tau)}\, d\tau$$

Changing DV 'τ' into 't'

$$\therefore FT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-(-j\omega)t}\, dt$$

$$\boxed{FT[x(-t)] = X(-j\omega)}$$

CONVOLUTION PROPERTY:

Stmt:-

If $FT[x_1(t)] = X_1(j\omega)$ &

$FT[x_2(t)] = X_2(j\omega)$

then

$$FT[x_1(t) * x_2(t)] = X_1(j\omega)\; X_2($$

Proof :-

w.k.T The convolution between two CT signal is given by

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

w.k.T

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\Rightarrow FT[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} [x_1(t) * x_2(t)]\, e^{-j\omega t}\, dt$$

let $(t-\tau) = T$ $\Rightarrow$ $t = T+\tau$

$dt = dT$

If $t = \infty$ then $T = \infty$

$t = -\infty$ then $T = -\infty$

$$① \Rightarrow FT[x_1(t) * x_2(t)] = \int_{\tau=-\infty}^{\infty} x_1(\tau)\, d\tau \int_{T=-\infty}^{\infty} x_2(T)\, e^{-j\omega(T+\tau)}\, dT$$

$$= \int_{\tau=-\infty}^{\infty} x_1(\tau)\, d\tau \int_{T=-\infty}^{\infty} x_2(T)\, e^{-j\omega T}\, e^{-j\omega\tau}\, dT$$

$$= \int_{\tau=-\infty}^{\infty} x_1(\tau)\, e^{-j\omega\tau}\, d\tau \int_{T=-\infty}^{\infty} x_2(T)\, e^{-j\omega T}\, dT$$

Changing DV T & τ into 't'

$$\therefore FT[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(t)\, e^{-j\omega t}\, dt \int_{-\infty}^{\infty} x_2(t)\, e^{-j\omega t}\, dt$$

$$\boxed{FT[x_1(t) * x_2(t)] = X_1(j\omega) \cdot X_2(j\omega)}$$

Problems:

1) Find the Fourier transform of the following signals & sketch the magnitude & phase as a function of frequency, including both positive & Negative frequencies.

(i) $\delta(t-5)$ (ii) $e^{-at}u(t)$

i) Given:

$$\delta(t-5)$$

To find :-

$$FT[\delta(t-5)] = ?$$ Amplitude & Phase spectrum = ?

Solution:

w.k.T $FT[x(t-t_0)] = e^{-j\omega t_0} X(j\omega)$

$$\Rightarrow FT[\delta(t-5)] = e^{-j\omega(5)} \Delta(j\omega)$$

w.k.T

$$\Delta(j\omega) = FT[\delta(t)] = 1$$

$$\therefore FT[\delta(t-5)] = e^{-5j\omega}$$

$(e^{-j\theta} = \cos\theta - j\sin\theta)$ $= \cos 5\omega - j\sin 5\omega$

Let $A = a + jb$

Magnitude, $|A| = \sqrt{a^2+b^2}$

phase, $\angle A = \tan^{-1}(b/a)$

Let $X(\omega) = \cos 5\omega - j\sin 5\omega$

$$|X(\omega)| = \sqrt{(\cos 5\omega)^2 + (\sin 5\omega)^2} = 1$$

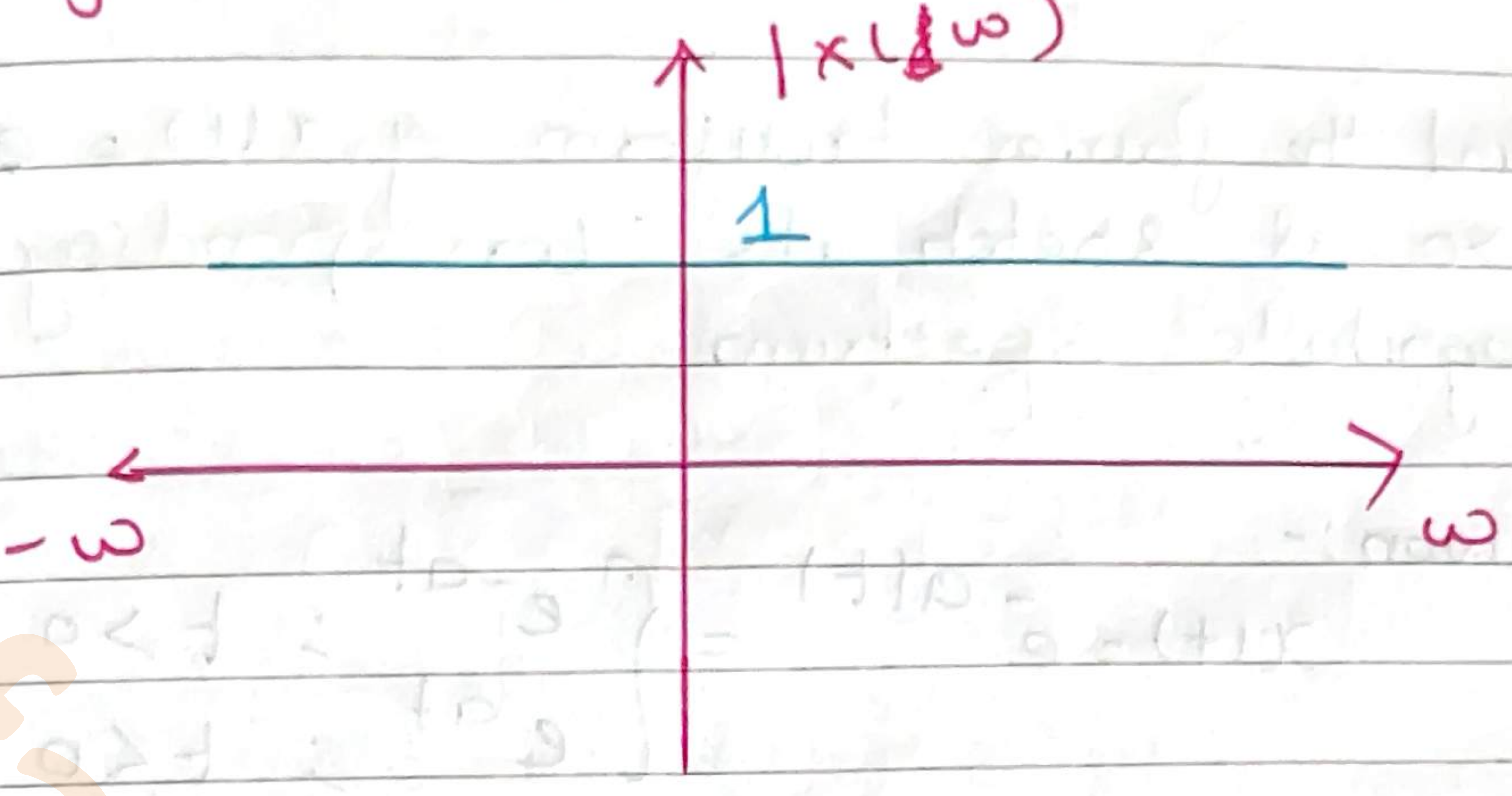
$$\angle x(\omega) = \tan^{-1}\left(\frac{\sin 5\omega}{\cos 5\omega}\right)$$

$$= \tan^{-1}(\tan 5\omega) = 5\omega$$

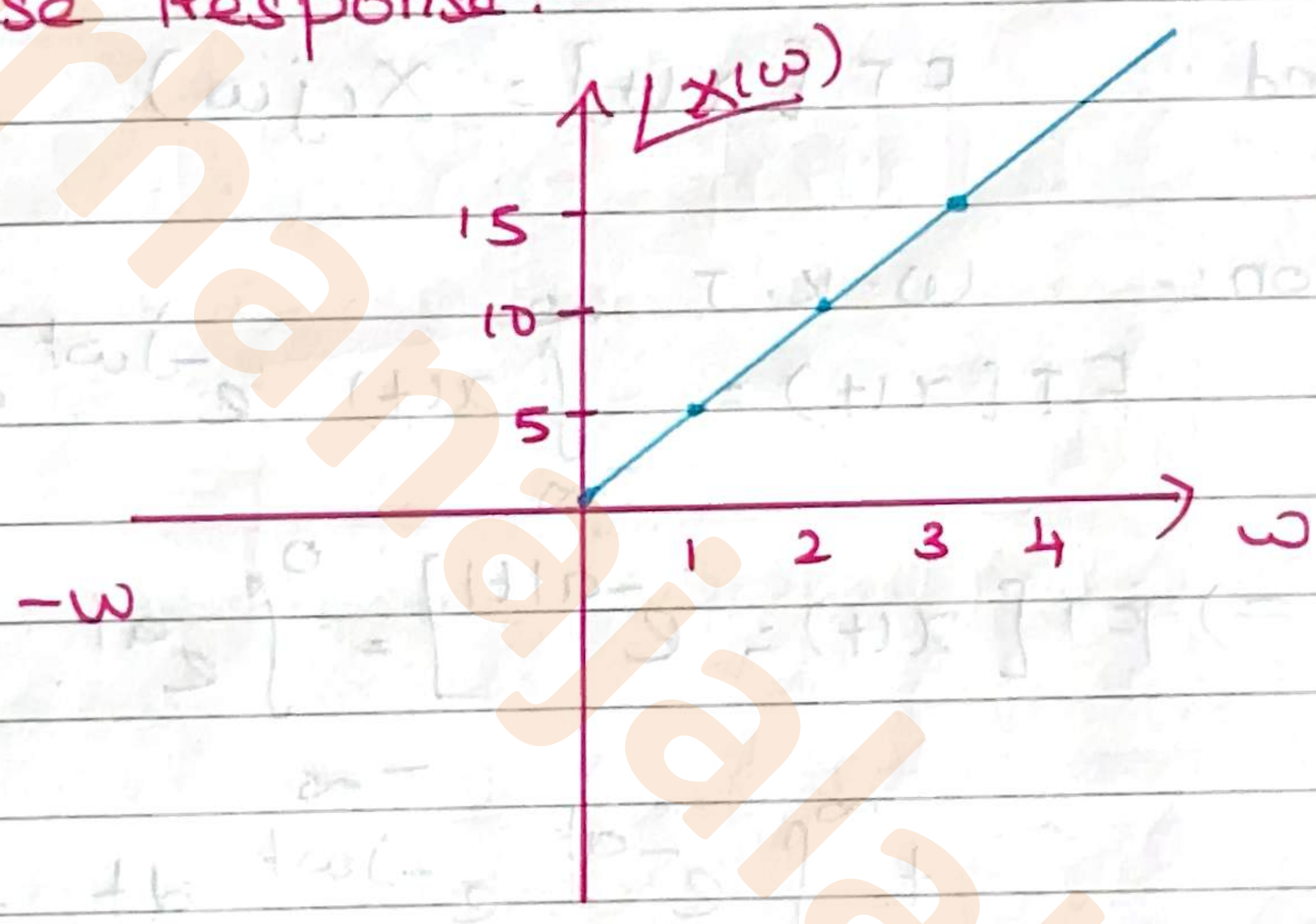
$\therefore$ Magnitude, $|X(\omega)| = 1$

Phase, $\angle x(\omega) = 5\omega$

Magnitude Response :-



Phase Response :-



ii) Given:

$$e^{-at}\,u(t)$$

To find

$$FT[e^{-at}\,u(t)] = ?$$

Solution:

w.k.T

$$FT[e^{-at}u(t)] = X(j\omega + a)$$

$$\Rightarrow FT[e^{-at}u(t) = U(j\omega)\Big|_{j\omega = j\omega + a}$$

$$X(\omega) = \frac{1}{j\omega + a}$$

magnitude, $|X(\omega)| = \left|\frac{1}{j\omega + a}\right|$

$$= \frac{\sqrt{1}}{\sqrt{a^2+\omega^2}} = \frac{1}{\sqrt{a^2+\omega^2}}$$

phase, $\angle X(\omega) = \tan^{-1}(1) - \tan^{-1}(\omega/a)$

Q2) Find the fourier transform of $x(t) = e^{-a|t|}$, $a>0$ & sketch it's corresponding magnitude spectrum.

Given:-

$$x(t) = e^{-a|t|} = \begin{cases} e^{-at} & ; t>0 \\ e^{at} & ; t<0 \end{cases}$$

To find: $FT[x(t)] = X(j\omega)$

Solution:- W.K.T

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\Rightarrow FT[x(t) = e^{-a|t|}] = \int_{-\infty}^{0} e^{at} \cdot e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{t(a-j\omega)}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \left[\frac{e^{(a-j\omega)t}}{a-j\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+j\omega)}}{-(a+j\omega)}\right]_{0}^{\infty}$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{a+\cancel{j\omega}+a-\cancel{j\omega}}{(a-j\omega)(a+j\omega)}$$

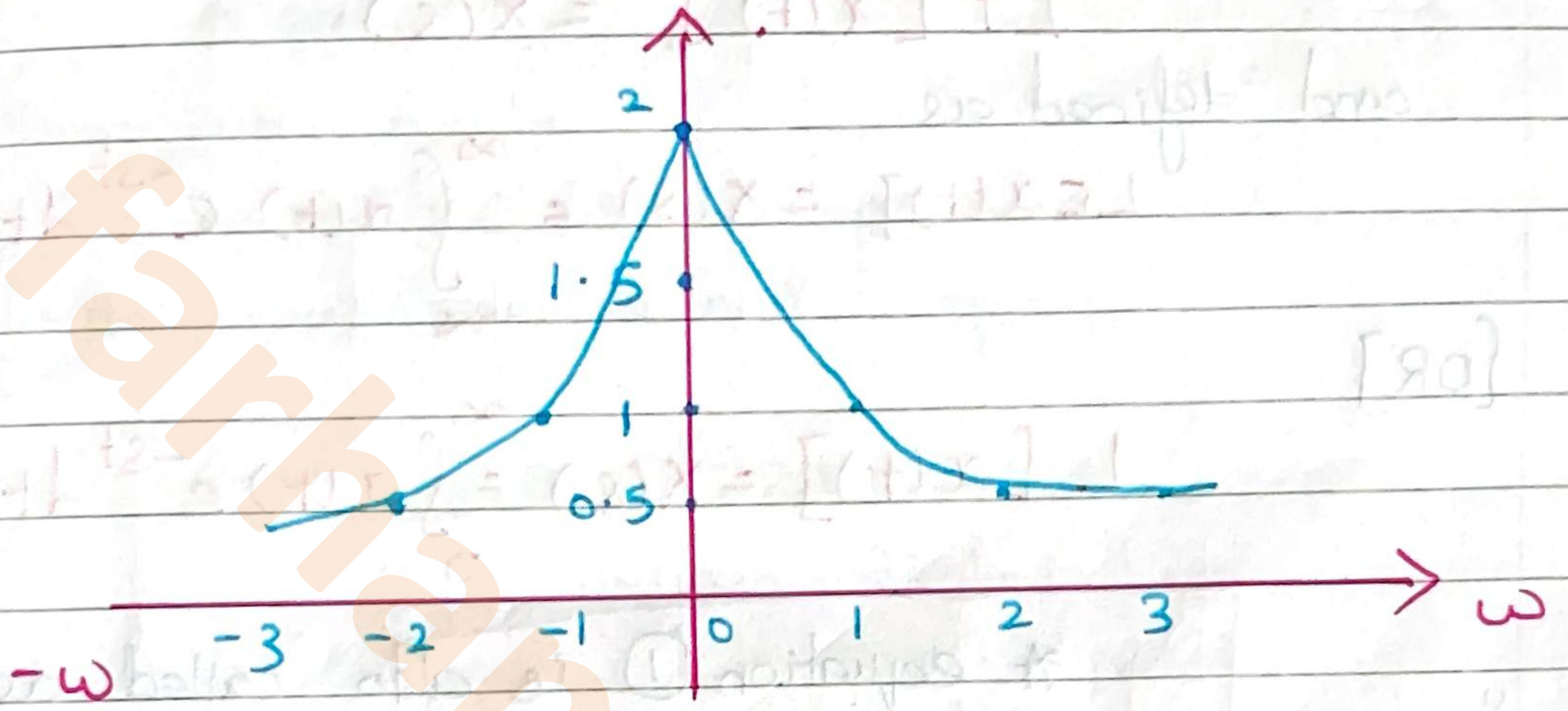
$$X(j\omega) = \frac{2a}{a^2+\omega^2}$$

Magnitude, $|X(j\omega)| = \left|\frac{2a}{a^2+\omega^2}\right|$

Let $a = 1 \Rightarrow |X(j\omega)| = \left|\frac{2}{1+\omega^2}\right|$

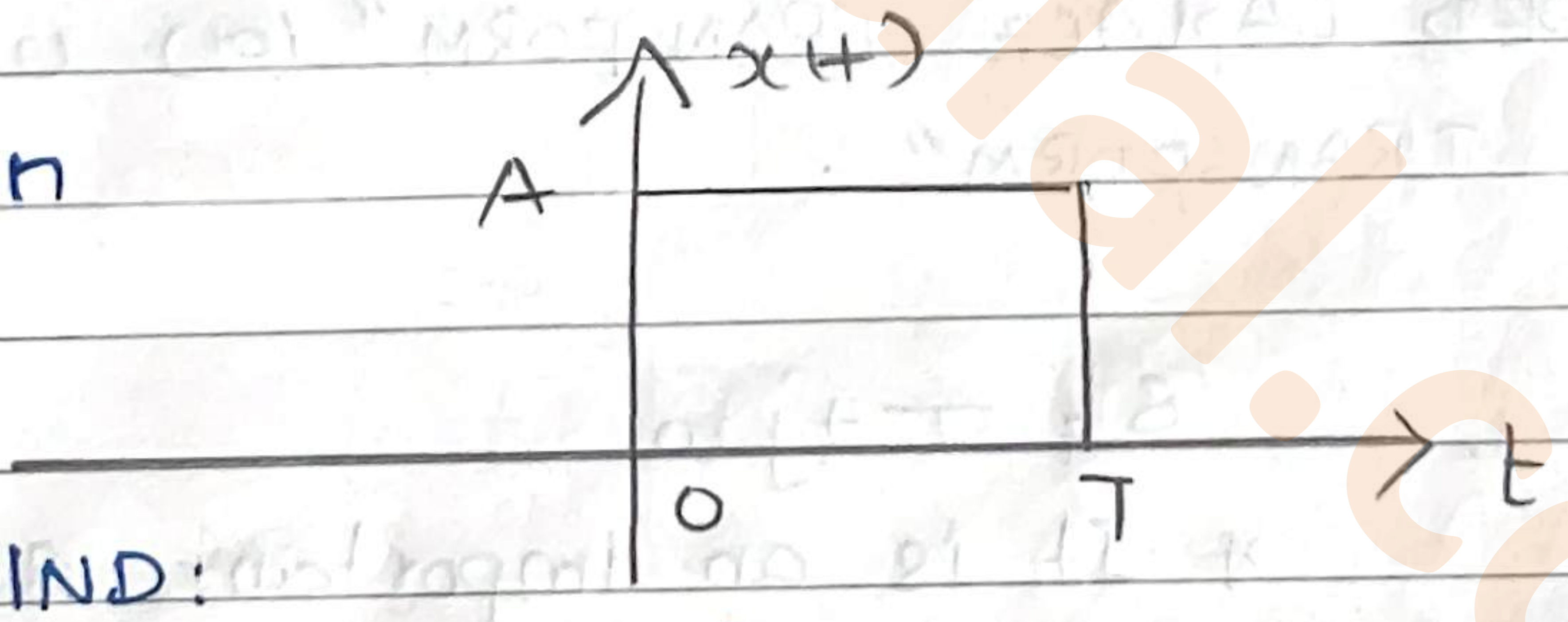
ω	-2	-1	0	1	2	3
$H(\omega)$	0.6	1	2	1	0.6	0.3

put $\omega = -2, -1, 0, 1, 2$



3] Find the fourier transform of a rectangular pulse with width T & amplitude 'A'

Given



$x(t) = A; \quad 0 \le t \le T$

$\quad\quad 0; \quad$ otherwise.

To FIND:

$$FT[x(t)] = X(j\omega) = ?$$

Solution:

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_0^T A\, e^{-j\omega t}\, dt$$

$$= A \int_0^T e^{-j\omega t}\, dt$$

$$= A\left[\frac{e^{-j\omega t}}{-j\omega}\right]_0^T = A\left\{\frac{e^{-j\omega t}}{-j\omega} - \frac{1}{j\omega}\right]$$

$$X(j\omega) = A\left[\frac{1 - e^{-j\omega t}}{j\omega}\right]$$

LAPLACE TRANSFORM:

* It is a mathematical tool for converting continuous time signal into continuous frequency.

* The Laplace tranform of any CT signal, $x(t)$ can be represented as

$$LT[x(t)] = X(s)$$

and defined as,

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \quad \text{--- (1)}$$

[OR]

$$L[x(t)] = X(s) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt \quad \text{--- (2)}$$

* equation (1) is also called as "TWO SIDED LAPLACE TRANSFORM" (or) BILATERAL LAPLACE TRANSFORM

* equation (2) is also called as "ONE SIDED LAPLACE TRANSFORM" (or) UNILATERAL LAPLACE TRANSFORM".

where

$$s = \sigma + j\omega$$

* It is an important & powerfull mathematical tool in the system analysis & design

* It is widely used for describing the continuous circuits and systems & also to analyse signal flow through the causal linear time invariant system with non-zero initial conditions.

REGION OF CONVERGENCE (ROC):

If $x(t)$ exsists for all time then $L[x(t)] = X(s)$ is also exsists for all 's' is called Region of convergence (ROC) ie;

As we know that for the existance of the Laplace transform, the integral

$$X(s) = \int_0^{\infty} x(t)\, e^{-st}\, dt$$

must converge, This limits the variable $s = \sigma + j\omega$ to be a part of the complex plane Known as the Region of convergence (ROC).

* ROC is required in computing the Laplace transform & the inverse Laplace transform.
* If ROC is not specified, the inverse Laplace transform is not unique.

INVERSE LAPLACE TRANSFORM:

✓ The inverse laplace transform is used to convert frequency domain [s-domain] signal $X(s)$ to the time domain signal $x(t)$

✓ It canbe represented as

$L^{-1}[X(s)] = x(t)$ and defined as,

$$x(t) = L^{-1}[X(s)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(s)\, e^{st}\, ds$$

PROBLEMS:

1) Find the Laplace transform of the following test signals.

(i) $u(t)$ (ii) $\delta(t)$ (iii) $r(t)$

(iv) e^{-at} (v) $\sin at$ (vi) $\cos at$

(i) Given:

Let $x(t) = u(t) = \begin{cases} 1 & ; t \geq 0 \\ 0 & ; t < 0 \end{cases}$

To find:

$L[x(t) = u(t)] = ?$

Page No.:
Date:
YOUVA

Solution:-

W.K.T $L[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$

$\Rightarrow L[u(t)] = \int_{-\infty}^{\infty} u(t)\, e^{-st}\, dt$

$= \int_{0}^{1} 1 \times e^{-st}\, dt$

$= \left[\frac{e^{-st}}{-s}\right]_0^1 \qquad e^{-\infty} = 0$

$= \frac{e^{-\infty}}{-s} - \frac{e^{0}}{-s}$

$$\boxed{L[u(t)] = U(s) = 1/s}$$

(ii) GIVEN:

Let $x(t) = \delta(t) = 1 \; ; \; t = 0$
$\qquad\qquad\qquad\quad 0 \; ; \; t \neq 0$

To find:

$L[\delta(t)] = \Delta(s) = ?$

Solution:

W.K.T $L[x(t)] = \int_{0}^{\infty} x(t)\, e^{-st}\, dt$

$\Rightarrow L[\delta(t)] = \int_{0}^{\infty} \delta(t)\, e^{-st}\, dt$

$= \left[\frac{e^{-st}}{-s}\right]_{t=0} = \delta(0) e^{-s(0)} = 1$

$$\boxed{L[\delta(t)] = \Delta(s) = 1}$$

(iii) Given:

Let $x(t) = r(t) = t \; ; t \geq 0$

$\qquad\qquad\qquad\quad 0 \; ; t < 0$

To find :

$L[x(t) = r(t)] = R(s) = ?$

Solution:

w.k.T $\quad L[x(t)] = \int_{-\infty}^{\infty} x(t) \, e^{-st} \, dt$

$\Rightarrow L[r(t)] = \int_{-\infty}^{\infty} r(t) \, e^{-st} \, dt$

$= \int_{0}^{\infty} t \, e^{-st} \, dt \quad \text{—— (1)}$

Let us consider,

$\int uv \, dx = u v_1 - u' v_2 + u'' v_3 - \ldots$

$u = t \quad ; \quad v \, dx = e^{-st} \, dt$

$u' = 1 \quad\quad v_1 = \int v \, dx = \int e^{-st} \, dt$

$u'' = 0 \quad\quad v_1 = \frac{e^{-st}}{-s}$

$v_2 = \frac{e^{-st}}{s^2}$

$(1) \Rightarrow L[r(t)] = \left[\frac{t \, e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^{\infty}$

$\therefore e^{-\infty} = 0$

$= \left[\infty \times \frac{e^{-\infty}}{-s} - \frac{e^{-\infty}}{s^2} \right] - \left(0 - \frac{e^0}{s^2} \right)$

$$\boxed{L[r(t)] = \frac{1}{s^2} = R(s)}$$

(iv) Given:

Let $x(t) = e^{-at} \; ; t \geq 0$

To find :

$L[x(t)] = ?$

Solution:

$L[x(t)] = \int_{-\infty}^{\infty} x(t) \, e^{-st} \, dt$

Page No.:
Date:

$$\Rightarrow L[e^{-at}] = \int_{-\infty}^{\infty} e^{-at} e^{-st} dt$$

$$= \int_{0}^{\infty} e^{-(s+a)t} dt$$

$$= \left[\frac{e^{-(s+a)}}{-(s+a)} \right]_0^{\infty}$$

$$= \left(\frac{e^{-\infty}}{-(s+a)} \right) - \left(\frac{e^{0}}{-(s+a)} \right)$$

$$\boxed{L[e^{-at}] = \frac{1}{s+a}}$$

v) Given:

Let $x(t) = \sin at \; ; t \geq 0$

To find:

$L[x(t) = \sin at] = ?$

Solution:

w.k.T $\quad L[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

$$L[\sin at] = \int_{0}^{\infty} \sin at \; e^{-st} dt$$

$$= \int_{0}^{\infty} e^{-st} \sin at \; dt \quad \text{— (1)}$$

w.k.T

$$\int e^{-ax} \sin bx \, dx = \frac{e^{-ax}}{a^2+b^2} (a \sin bx + b \cos bx) \quad \text{— (2)}$$

using eqn (2) in (1) $\quad a = s; b = a; x = t$

$$(1) \Rightarrow L[\sin at] = \left[\frac{e^{-at}}{s^2+a^2} (s \sin at - a \cos at) \right]_0^{\infty}$$

$$= \left\{ \frac{e^{-\infty}}{s^2+a^2} (\cancel{s \sin \infty} - a \cos \infty) - \frac{e^{0}}{s^2+a^2} (a \sin(0) - a \cos 0) \right.$$

$$\boxed{L\,[\sin at] = \frac{a}{s^2+a^2}}$$

(2) Find the Laplace transform of $x(t) = e^{-a(t)}$; $a>0$ & its associated ROC.

Given:

$$x(t) = e^{-a|t|} \quad ; \quad a>0$$

ie);

$$x(t) = \begin{cases} e^{-at} & ; t>0 \\ e^{+at} & ; t<0 \end{cases}$$

To find:

$L[x(t)]$ & It's ROC

Solution:

w.k.T

$$L[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{0} e^{at} e^{-st}\, dt \int_{0}^{\infty} e^{-at} e^{-st}\, dt$$

$$= \int_{-\infty}^{0} e^{(a-s)t}\, dt + \int_{0}^{\infty} e^{-(s+a)t}\, dt$$

$$= \left[\frac{e^{(a-s)t}}{(a-s)}\right]_{-\infty}^{0} + \left[\frac{e^{-(s+a)t}}{(s+a)}\right]_{0}^{\infty}$$

$e^0 = 1$

$\therefore e^{-\infty} = 0$

$$\Rightarrow \left[\frac{e^0}{(a-s)} - \frac{e^{-\infty}}{(a-s)}\right] + \left[\frac{e^{-\infty}}{(s+a)} - \frac{e^0}{(s+a)}\right]$$

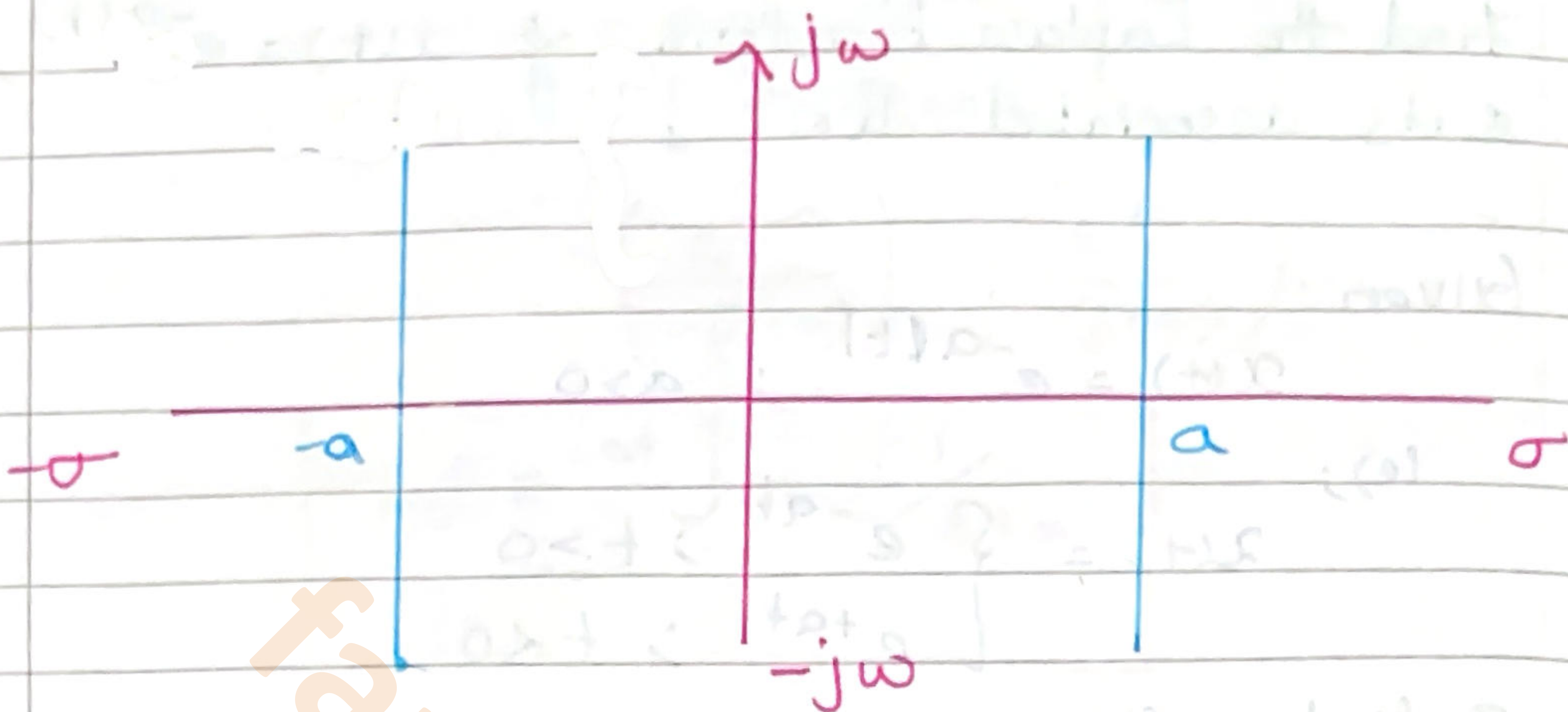
$$= \frac{1}{(a-s)} - \frac{1}{(s+a)}$$

$\because a^2-b^2 = (a+b)(a-b)$

$$= \frac{(a-s)-(s+a)}{s^2-a^2}$$

$$L[x(t)] = \frac{-2a}{s^2-a^2}$$

ROC: $|s| > a$ & $|s| < a$



(2) Find the Laplace transform of the signal $x(t) = e^{-2t}u(t) + e^{-t}\cos 3t$

Given:

$$x(t) = e^{-2t}u(t) + e^{-t}\cos 3t \quad \text{—} (1)$$

To FIND:

$$L[x(t)] = X(s) = ?$$

Solution:

w.k.T

$$L[e^{-at}x(t)] = X(s)\Big|_{s \to (s+a)}$$

on taking LT bothsides in eqn (1)

$$(1) \Rightarrow L[x(t)] = L[e^{-2t}u(t) + e^{-t}\cos 3t]$$

$$= L[e^{-2t}u(t)] + L[e^{-t}\cos 3t]$$

$\therefore L[u(t)] = 1/s$

$L[\cos at] = a/(s^2+a^2)$

$$= L[u(t)]\Big|_{s \to s+2} + L[\cos 3t]\Big|_{s \to (s+1)}$$

$$X(s) = \frac{1}{s+2} + \frac{3}{(s+1)^2+9}$$

4) Find the inverse laplace transform of $X(s)$

$$X(s) = \frac{3s^2+8s+6}{(s+2)(s^2+2s+1)}$$

Given:

$$X(s) = \frac{3s^2+8s+6}{(s+2)(s^2+2s+1)}$$

To find:

$$L^{-1}[X(s)] = x(t) = ?$$

using partial fraction Method

Solution:

Let $$\frac{3s^2+8s+6}{(s+2)\underbrace{(s^2+2s+1)}_{\text{Roots of s-poly}}} = \frac{3s^2+8s+6}{(s+2)(s+1)^2}$$

$$= \frac{A}{s+2} + \frac{B}{(s+1)} + \frac{C}{(s+1)^2}$$

$$\frac{3s^2+8s+6}{\cancel{(s+2)(s+1)^2}} = \frac{A(s+1)^2 + B(s+1)(s+2) + C(s+2)}{\cancel{(s+2)(s+1)^2}}$$

$$3s^2+8s+6 = A(s+1)^2 + B(s+1)(s+2) + C(s+2) \quad \text{--- (2)}$$

Put $s = -2$

$$\Rightarrow 3(-2)^2 + 8(-2) + 6 = A(-2+1)^2$$

$$12 - 16 + 6 = A$$

$$\boxed{A = 2}$$

Put $s = -1$ in eqn (2)

$$3(-1)^2 + 8(-1) + 6 = C(-1+2)$$

$$3 - 8 + 6 = C$$

$$\boxed{C = 1}$$

Put $S=0$ in eqn (2)

$$6 = A + 2B + 2C$$

$$\Rightarrow 2B = 6 - A - 2C$$

$= 6 - 2 - 2$

$2B = 2$

$\boxed{B = 1}$

$$① \Rightarrow X(s) = \frac{2}{s+2} + \frac{1}{s+1} + \frac{1}{(s+1)^2}$$

ie;

$$L[x(t)] = \frac{2}{s+2} + \frac{1}{s+1} + \frac{1}{(s+1)^2}$$

$$\Rightarrow x(t) = L^{-1}\left[\frac{2}{s+2}\right] + L^{-1}\left[\frac{1}{s+1}\right] + L^{-1}\left[\frac{1}{(s+1)^2}\right]$$

$$x(t) = 2e^{-2t}u(t) + e^{-t}u(t) + t\,e^{-t}u(t)$$

$$\because L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}u(t)$$

$$L^{-1}\left[\frac{1}{(s+a)^2}\right] = t\,e^{-at}u(t)$$

5] Determine the inverse laplace transform of the following:

(i) $X(s) = \dfrac{1 - 2s^2 - 14s}{s(s+3)(s+4)}$

(ii) $X(s) = \dfrac{2s^2 + 10s + 7}{(s+1)(s^2+3s+2)}$

(i) Given:

$$X(s) = \frac{1 - 2s^2 - 14s}{s(s+3)(s+4)} \quad \text{——} \quad ①$$

To find:

$$L^{-1}[X(s)] = x(t) = ?$$

using partial fraction method.

let $\dfrac{1 - 2s^2 - 14s}{s(s+3)(s+4)} = \dfrac{A}{s} + \dfrac{B}{s+3} + \dfrac{C}{s+4}$

$s = 0 \quad s = -3 \quad s = -4$

$$\frac{1-2s^2-14s}{s(s+3)(s+4)} = \frac{A(s+3)(s+4)+B(s)(s+4)+C(s)(s+3)}{s(s+3)(s+4)}$$

$$\Rightarrow 1-2s^2-14s = A(s+3)(s+4)+B(s)(s+4)+C(s)(s+3) \quad \text{—— (2)}$$

put $s=0$ in eqn (2)

$$1 = A(3)(4) = 12A$$

$$\boxed{A = 1/12}$$

put $s=-3$ in eqn (2)

$$1-2(-3)^2-14(-3) = B(-3)(-3+4)$$

$$1-18+42 = -3B$$

$$25 = -3B$$

$$\boxed{B = -25/3}$$

put $s=-4$ in eqn (2)

$$1-2(-4)^2-14(-4) = C(-4)(-4+3)$$

$$1-32+56 = 4C$$

$$25 = 4C$$

$$\boxed{C = 25/4}$$

(1) $\Rightarrow X(s) = \frac{(1/12)}{s} + \frac{(-25/3)}{s+3} + \frac{(25/4)}{s+4}$

ie) $L[x(t)] = \frac{1}{12s} - \frac{25}{3(s+3)} + \frac{25}{4(s+4)}$

$$\Rightarrow x(t) = \frac{1}{12}L^{-1}\left[\frac{1}{s}\right] - \frac{25}{3}L^{-1}\left[\frac{1}{s+3}\right] + \frac{25}{4}L^{-1}\left[\frac{1}{s+4}\right]$$

$$= 0.083u(t) - 8.33e^{-3t}u(t) + 6.25e^{-4t}u(t)$$

$\because L^{-1}\left[\frac{1}{s}\right] = u(t)$

$L^{-1}\left[\frac{1}{s+3}\right] = e^{-at}u(t)$

Important Formula:-

$L[u(t)] = 1/s$

$L[\delta(t)] = 1$

$L[r(t)] = 1/s^2$

$L[e^{-at}u(t)] = 1/s+a$

$L[e^{at}u(t)] = 1/s-a$

$L[\sin at\, u(t)] = a/s^2+a^2$

$L[\sinh at\, u(t)] = a/s^2-a^2$

$L[\cos at\, u(t)] = \frac{s}{s^2+a^2}$

$L[\cosh at\, u(t)] = \frac{s}{s^2-a^2}$

$L[K] = K/s$

$L[t^n] = n!/s^{n+1}$

$L^{-1}[1/s] = u(t)$

$L^{-1}[1] = \delta(t)$

$L^{-1}[1/s^2] = r(t)$

$L^{-1}[\frac{1}{s+a}] = e^{-at}u(t)$

$L^{-1}[1/s-a] = e^{at}u(t)$

$L^{-1}[a/s^2+a^2] = \sin at\, u(t)$

$L^{-1}[a/s^2-a^2] = \sinh at\, u(t)$

$L^{-1}[s/s^2+a^2] = \cos at\, u(t)$

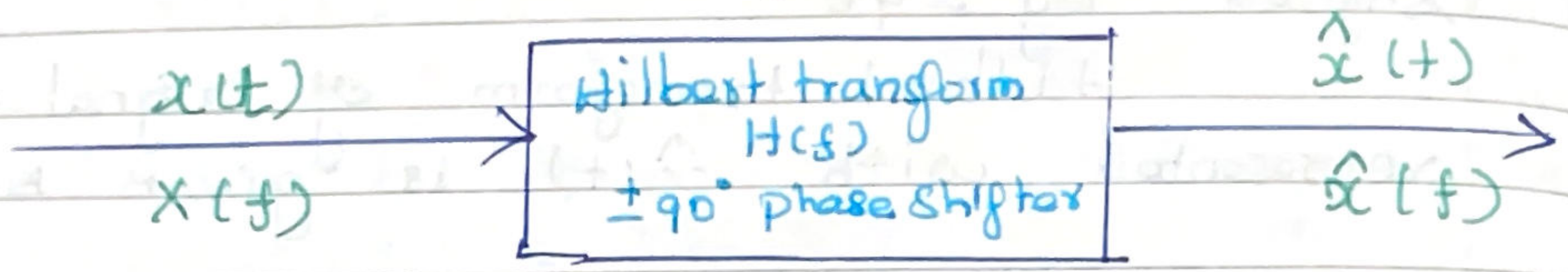
$L^{-1}[\frac{s}{s^2-a^2}] = \cosh at\, u(t)$

$L^{-1}[K/s] = K.$

$L^{-1}[n!/s^{n+1}] = t^n$

HILBERT TRANSFORM :

* Hilbert transform is a linear operator
* It takes the signal (function) & produce an another signal in the same domain



✓ The hilbert transform $\hat{x}(t)$ of a signal $x(t)$ is given by

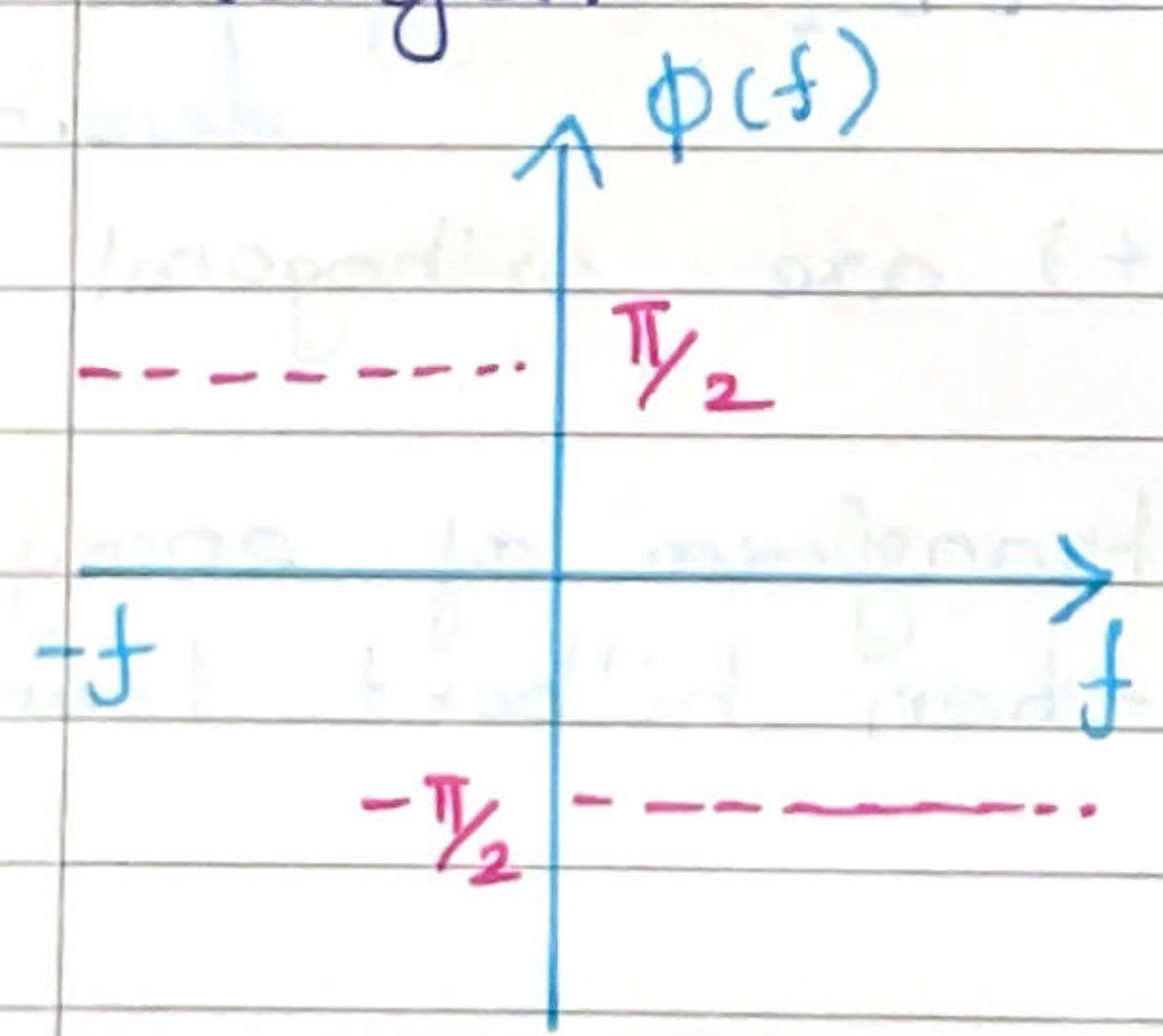
$$\boxed{\hat{x}(t) = x_h(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau}\, d\tau}$$

$$= x(t) * \frac{1}{\pi t}$$

✓ both the signals are in the time domain.

Characteristics :-

* Amplitude doesn't change in the hilbert transform.



→ Phase of the positive frequency components is shifted by $-\pi/2$ degree

→ Phase of the negative frequency components is shifted by $+\pi/2$ degree.

Properties of hilbert transform :

* It is a linear function.

$$H[a x_1(t) + b x_2(t)] = aH[x_1(t)] + bH[x_2(t)]$$

$$= a\hat{x}_1(t) + b\hat{x}_2(t)$$

* Time dilation [Time scaling]

$$H[x(at)] = \text{sgn}(a)\, \hat{x}(at) \quad ; a \neq 0$$

* Time derivative property :-

$$H\left[\frac{d}{dt} x(t)\right] = \frac{d}{dt} H[x(t)] = \frac{d}{dt} \hat{x}(t)$$

* convolution :-

$$H[x_1(t) * x_2(t)] = \hat{x}_1(t) * x_2(t)$$

(or)

$$x_1(t) * \hat{x}_2(t)$$

* A signal $x(t)$ & its hilbert transform $\hat{x}(t)$ both have the same energy density spectrum & same auto correlation

* $x(t)$ & $\hat{x}(t)$ both are Mutually orthogonal

$$\int_{-\infty}^{\infty} x(t) \cdot \hat{x}_h(t)\, dt = 0 .$$

* If $H[x(t)] = \hat{x}(t)$, then

$$H[\hat{x}(t)] = -x(t)$$

The inverse hilbert transform :-

$$x(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{x}(s)}{t-s}\, ds .$$

Signals & Its Hilbert Transform :-

Signal $x(t)$	Hilbert Transform $\hat{x}(t)$
$u(t)$	$H(u(t))$
$\sin t$	$-\cos t$
$\cos t$	$\sin t$

Signal $x(t)$	Hilbert transform $\hat{x}(t)$
$\Pi(t)$	$\frac{1}{\pi}\log\left\lvert\frac{t+1/2}{t-1/2}\right\rvert$
$\delta(t)$	$\frac{1}{\pi t}$
$x(a-b)t$	$\frac{2}{\pi}\log\left\lvert\frac{t-a}{t-b}\right\rvert$

The fourier transform of the hilbert Transform:-

$$\hat{X}(f) = -j\,\mathrm{sgn}(f)\,X(f)$$

where,

$$X(f) = F.T\{x(t)\}$$

Problems:-

1) Find the hilbert Transform & following signals.

i) $\delta(t)$ ii) $\sin\omega_0 t$ iii) $\cos\omega_0 t$

i) Solution:-

Given: $\delta(t)$.

w.k.T

$$\hat{x}(t) = x_h(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(\tau)}{x(t-\tau)}\,d\tau$$

$$= x(t) * \frac{1}{\pi t}$$

$$= \delta(t) * \frac{1}{\pi t}$$

Sub $t = T$ $t = t - T)$

$$= \int_{-\infty}^{\infty}\delta(T)\frac{1}{\pi(t-T)}\,dT$$

Sub $T = 0$

$$\therefore \delta(t) = \frac{1}{\pi t}$$