

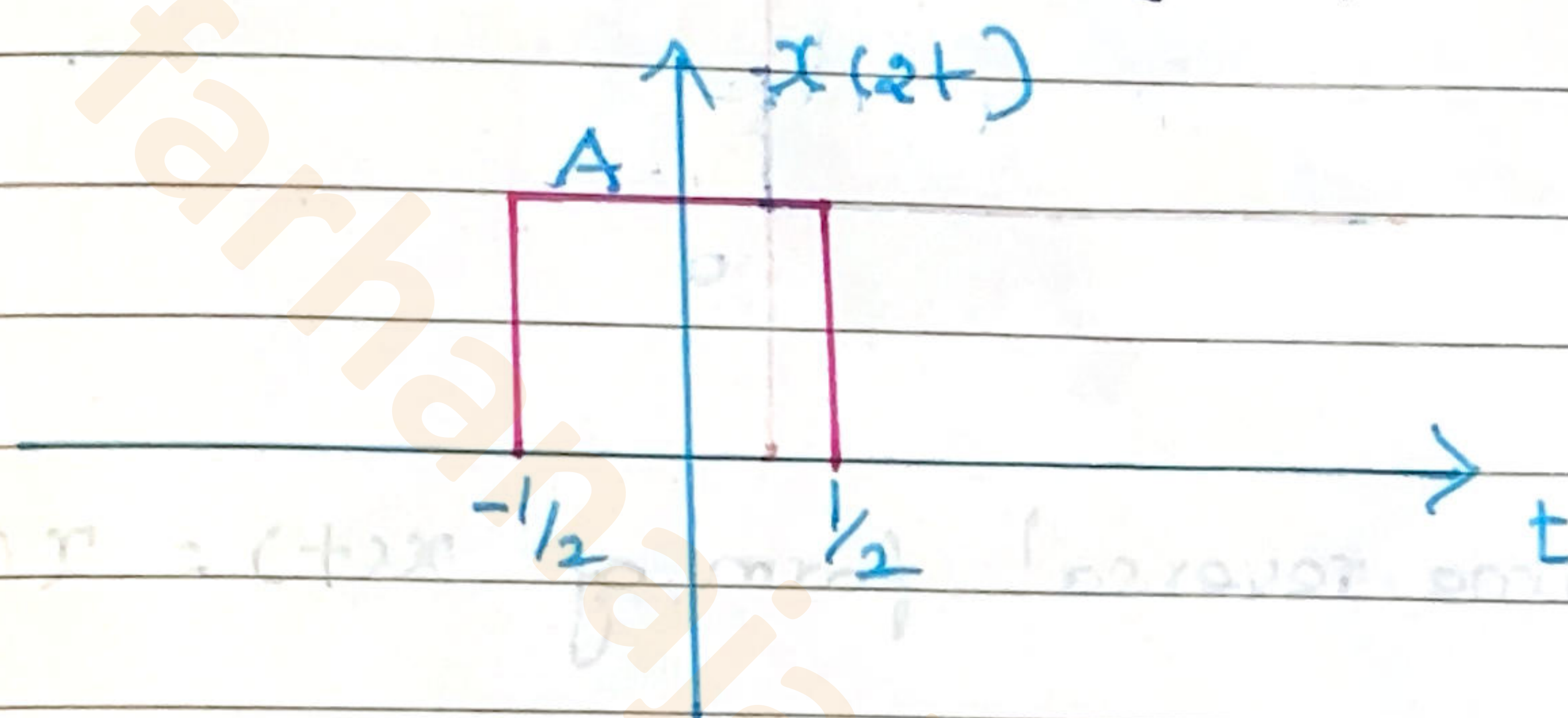
Given Rectangular signal

$$x(t) = A \cdot \text{rect}\left(\frac{t}{2}\right) = \begin{cases} A, & -1 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

* Time scaling (compressed) form of $x(t)$,

$$x(2t) = A \cdot \text{rect}\left(\frac{2t}{2}\right) =$$

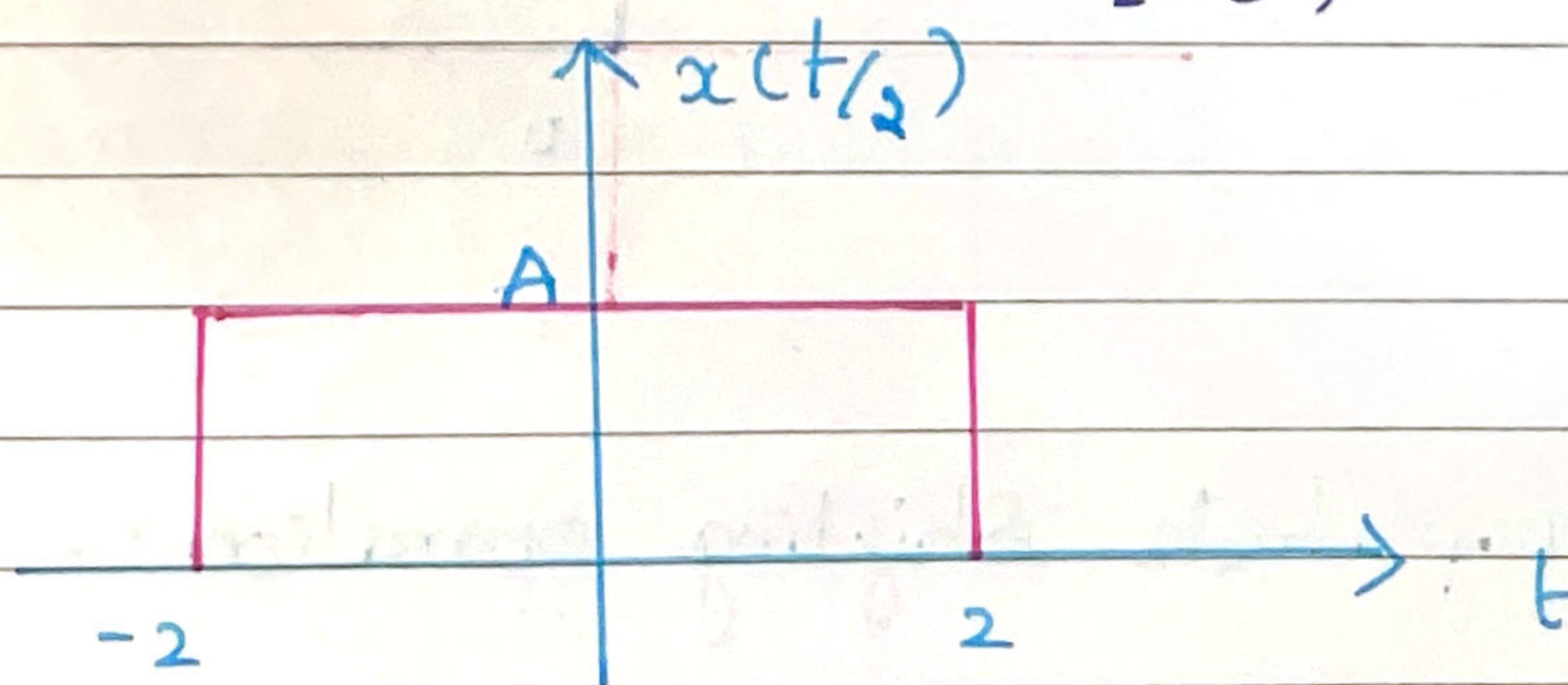
$$A \cdot \text{rect}\left(\frac{t}{1}\right) = \begin{cases} A, & -\frac{1}{2} < t < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$



* Time scaling (Expanded) form of $x(t)$,

$$x\left(\frac{t}{2}\right) = A \cdot \text{rect}\left(\frac{t/2}{2}\right)$$

$$= A \cdot \text{rect}\left(\frac{t}{4}\right) = \begin{cases} A, & -2 < t < 2 \\ 0, & \text{otherwise} \end{cases}$$



Time Reversal (or) Folding Operation:-

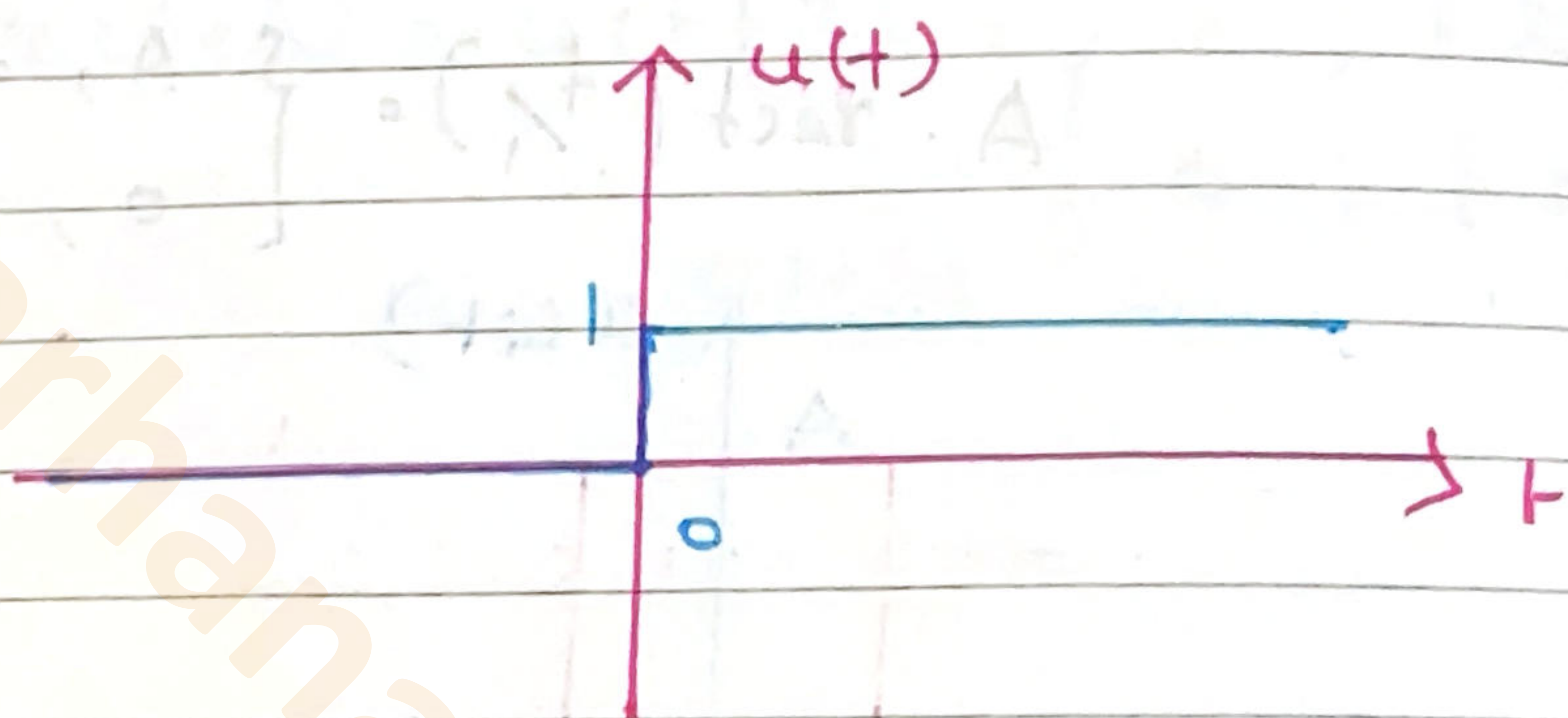
Time reversal signal can be obtained by interchanging left and right hand side positions with respect to Vertical axis (or) y-axis. If the given

Signal is $x(t)$, then its time reversal form is represented with $x(-t)$ and the operation is called folding (or) mirror image (or) time reversal.

Eg :

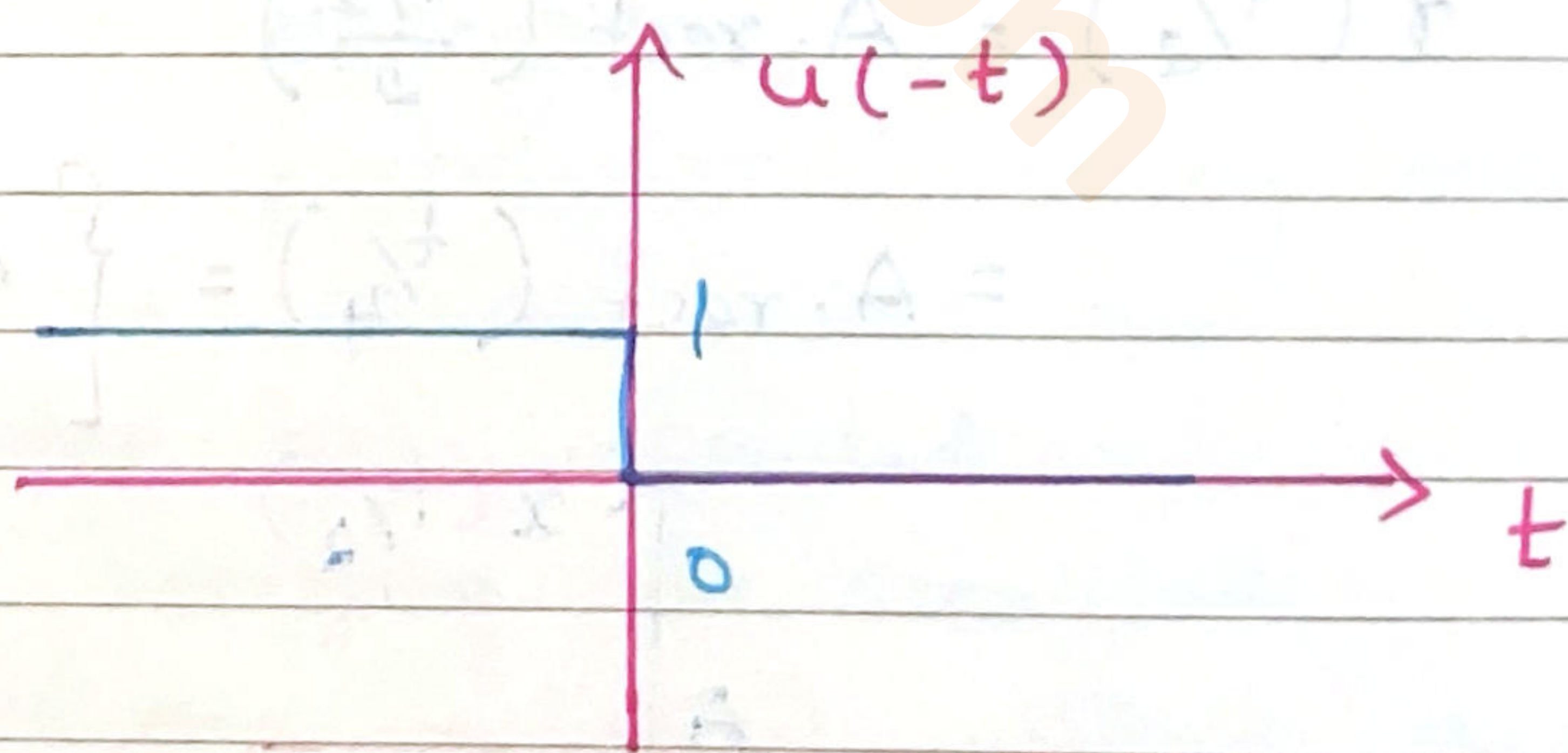
Given: unit step signal $x(t) = u(t)$

$$u(t) = \begin{cases} 1 & ; t \geq 0 \\ 0 & ; t < 0 \end{cases}$$



Time reversal form of $x(t) = x(-t)$

$$x(-t) = u(-t) = \begin{cases} 1 & ; t \leq 0 \\ 0 & ; t > 0 \end{cases}$$



Amplitude Shifting operation:-

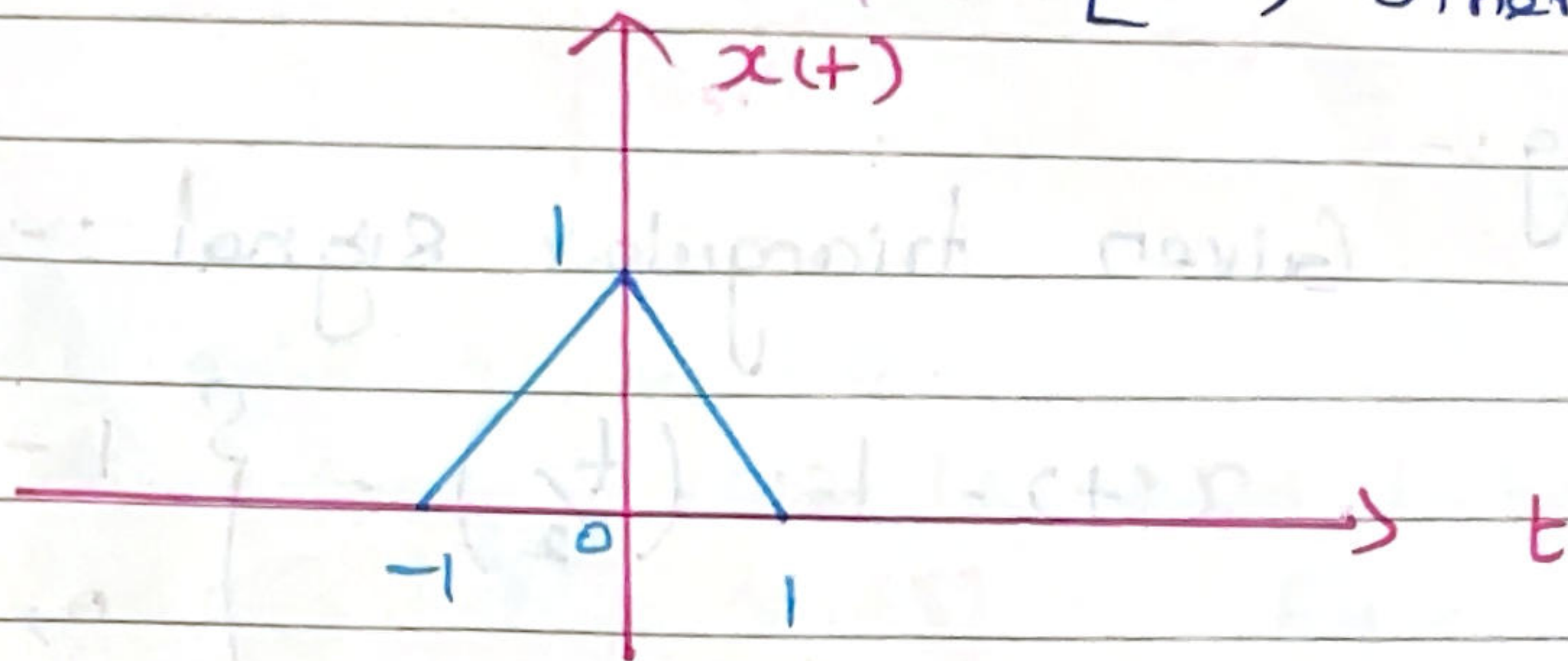
If amplitude shifting operation is applied on a signal $x(t)$, then the signal shifted to up (or) down.

It is represented as $A + x(t)$ where A is constant. It is also known as amplitude (or) DC level shifter.

Eg:-

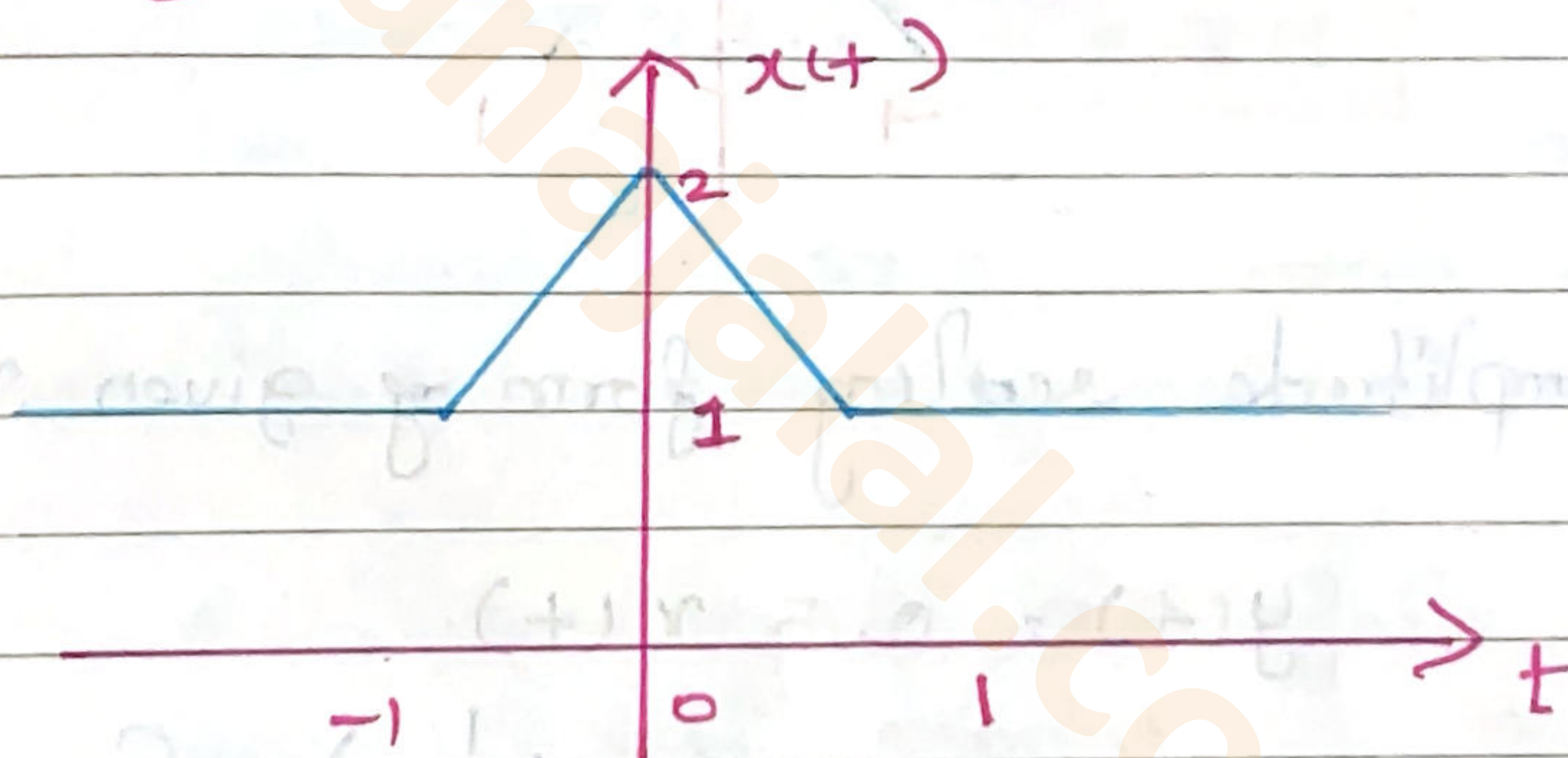
Given Triangular signal.

$$x(t) = \text{tri}(t/2) = \begin{cases} 1 - |t| & ; -1 < t < 1 \\ 0 & , \text{ otherwise.} \end{cases}$$



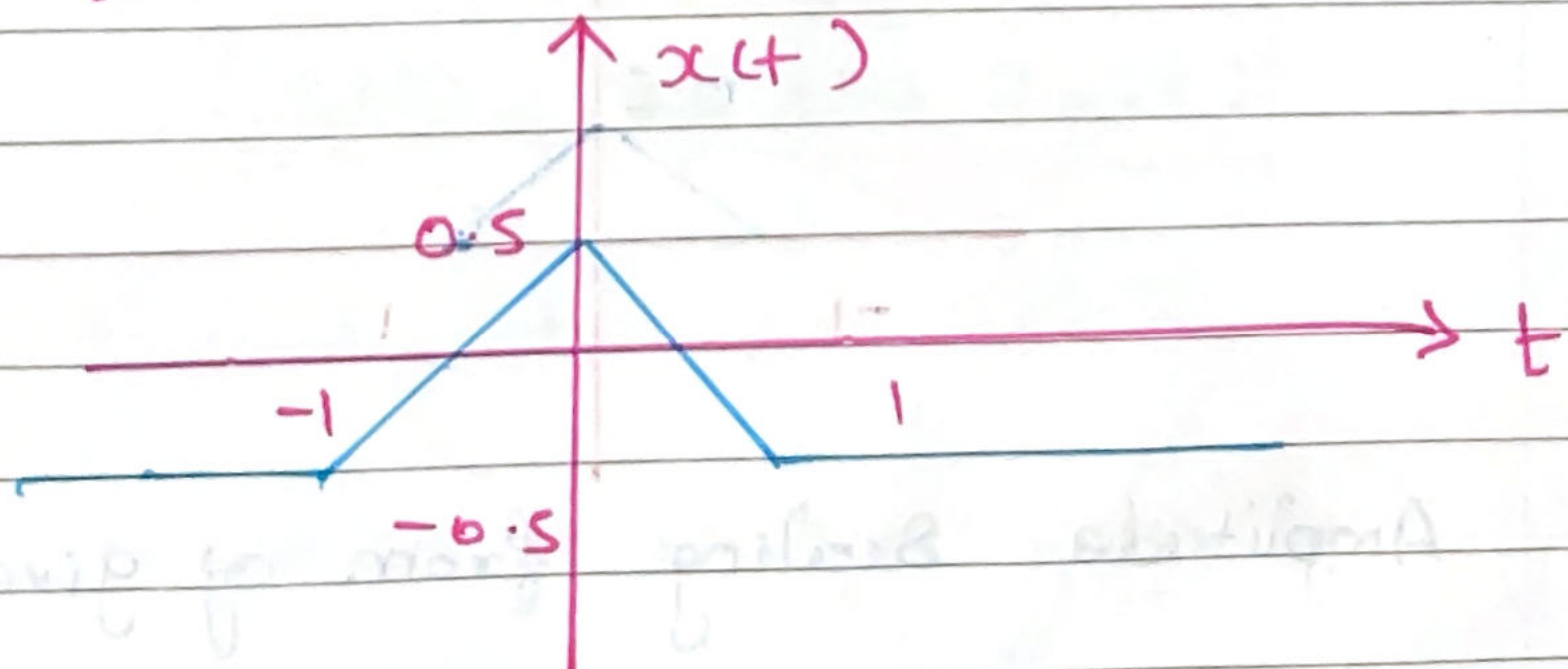
Amplitude shifting of $x(t)$,

$$y(t) = 1 + x(t)$$



Amplitude shifting of $x(t)$,

$$y(t) = -0.5 + x(t)$$



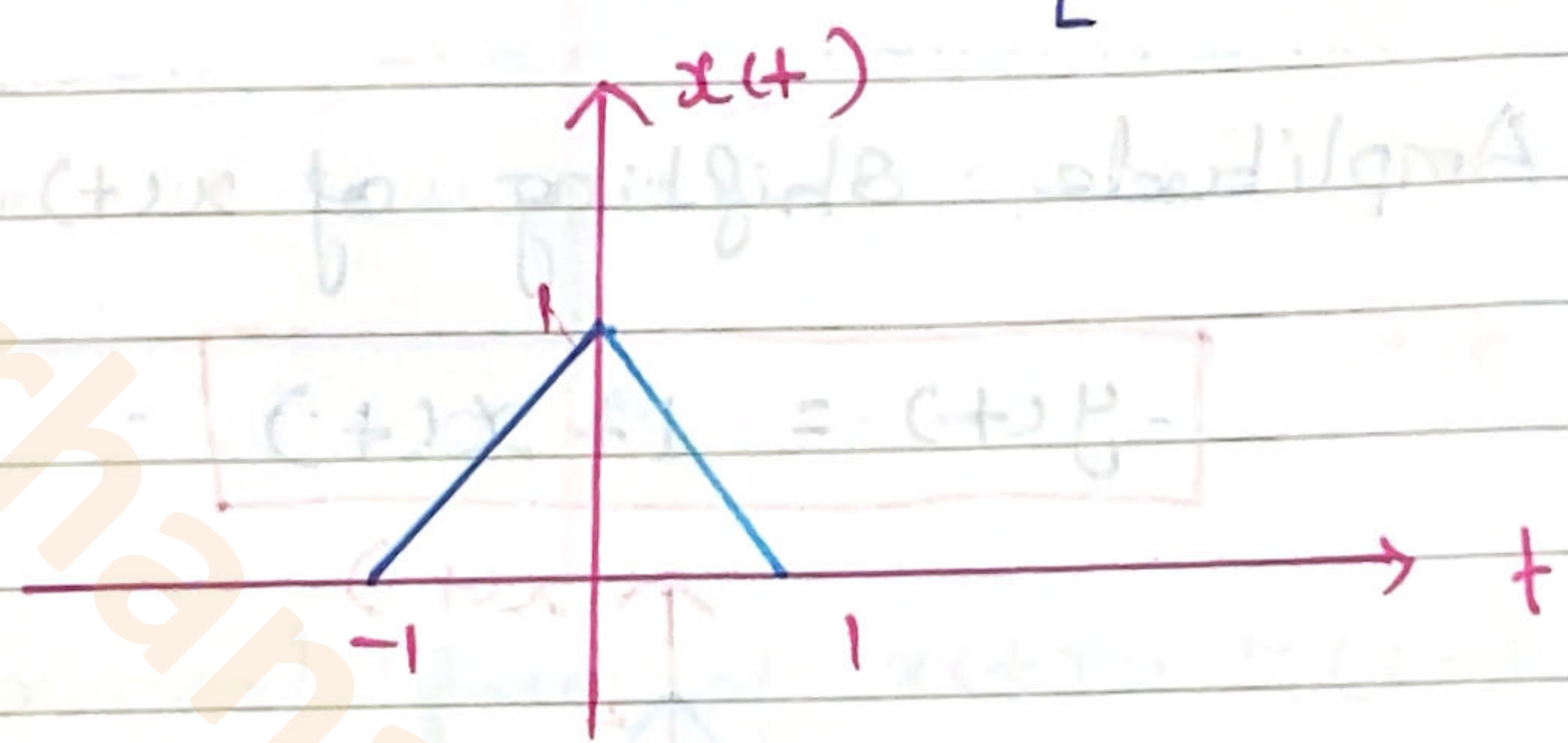
Amplitude scaling operation:-

If the amplitude scaling operation is applied on a signal $x(t)$

Then the signal amplitude may increase (or) decrease without changing its duration. It is represented with $Ax(t)$ where A is amplitude scaling parameter.

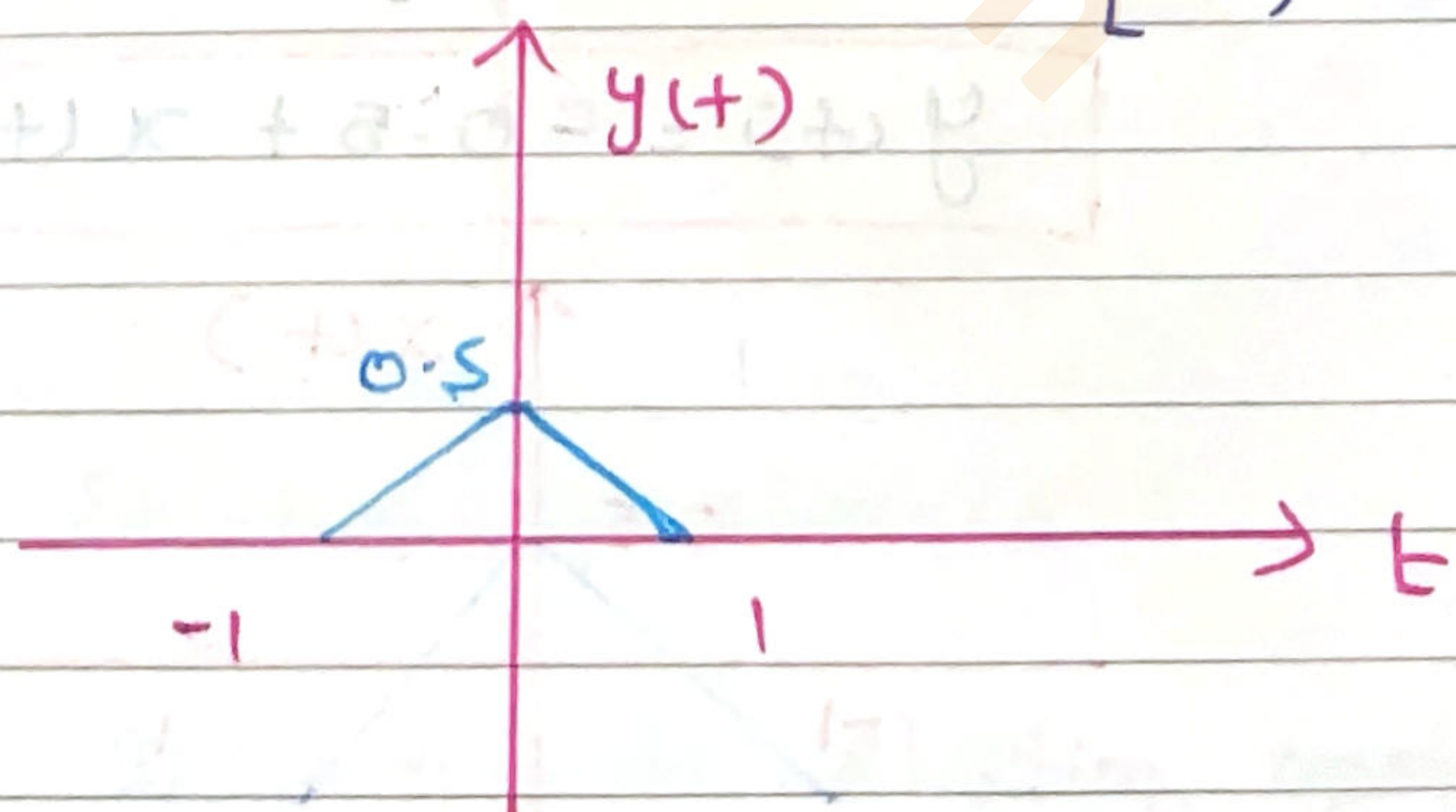
Eg:- Given triangular signal :-

$$x(t) = \text{tri}\left(\frac{t}{2}\right) = \begin{cases} 1 - |t|, & -1 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$



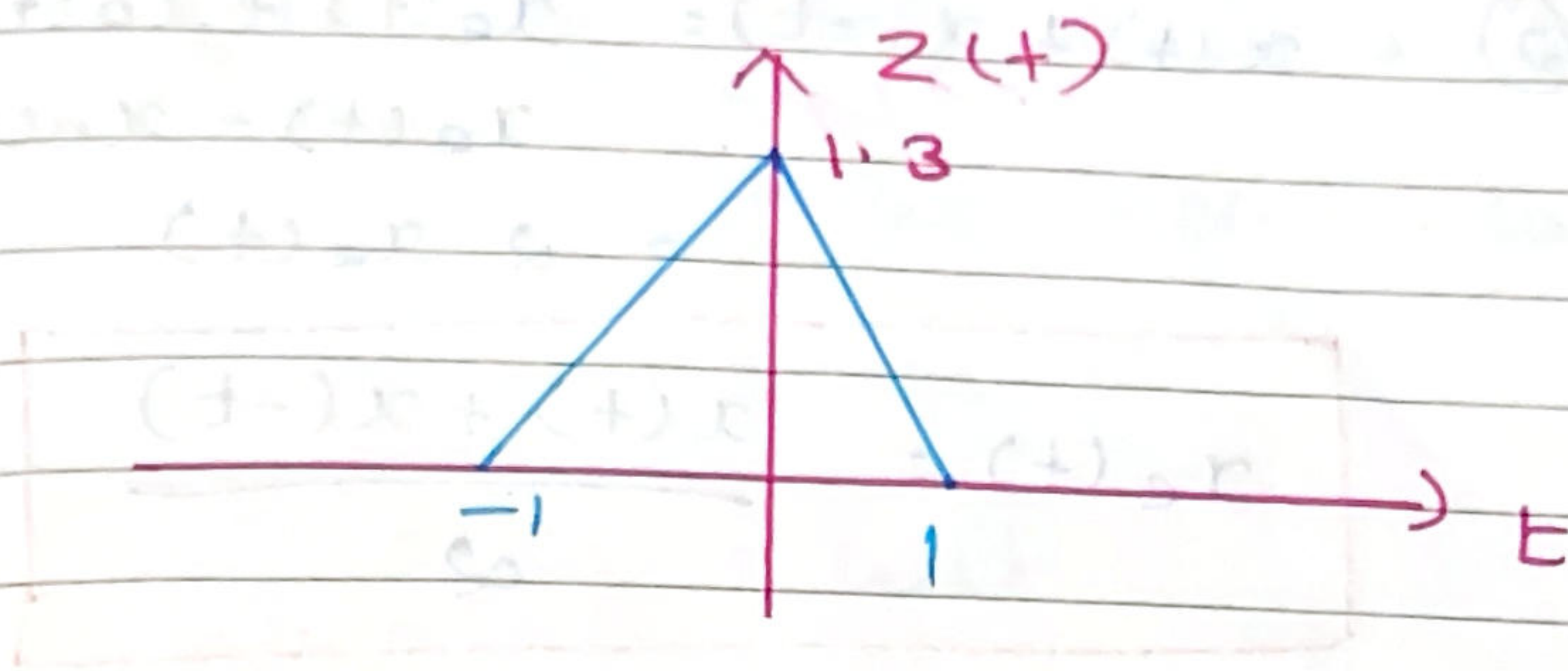
Amplitude scaling form of given signal is

$$\begin{aligned} y(t) &= 0.5 x(t) \\ &= 0.5 \text{tri}\left(\frac{t}{2}\right) = \begin{cases} 0.5(1 - |t|); & -1 < t < 1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$



Amplitude Scaling form of given signal.

$$\begin{aligned} z(t) &= 1.3 x(t) \\ &= 1.3 \text{tri}\left(\frac{t}{2}\right) = \begin{cases} 1.3(1 - |t|); & -1 < t < 1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$



Even and odd signal :-

* A signal $x(t)$ is said to be even only when $x(-t) = x(t)$. Even signals are symmetrical about y-axis, hence even signals are called symmetry signals.

* A signal $x(t)$ is said to be odd only when $x(-t) = -x(t)$. Odd signals are anti-symmetrical about y-axis, hence odd signals are called anti-symmetrical signals.

* If the signal $x(t)$ fails to satisfy even and odd property, then the signal is neither odd nor even and it can be expressed as the sum of even signal $x_e(t)$ and odd signal $x_o(t)$.

$$x(t) = x_e(t) + x_o(t) \quad \text{--- (1)}$$

Replace 't' with '-t'

$$x(-t) = x_e(-t) + x_o(-t)$$

$$\Rightarrow x(-t) = x_e(t) - x_o(t) \quad \text{--- (2)}$$

Where $x_e(t)$ = even part of $x(t)$

$x_o(t)$ = odd part of $x(t)$.

$$\textcircled{1} + \textcircled{2} \quad x(t) + x(-t) = x_e(t) + x_o(t) + x_e(t) - x_o(t) = 2x_e(t)$$

$$x_e(t) = \frac{x(t) + x(-t)}{2}$$

$$\textcircled{1} - \textcircled{2} \Rightarrow x(t) - x(-t) = x_e(t) + x_o(t) - (x_e(t) - x_o(t)) = 2x_o(t)$$

$$x_o(t) = \frac{x(t) - x(-t)}{2}$$

Eg:-

$$x(t) = \cos t + \sin t + \cos t \sin t$$

1. Find the even and odd components

Given:-

$$x(t) = \cos t + \sin t + \cos t \sin t$$

$$\text{W.K.T :- } x_e(t) = \frac{1}{2}(x(t) + x(-t))$$

$$x_o(t) = \frac{1}{2}(x(t) - x(-t))$$

$$x(-t) = \cos(-t) + \sin(-t) + \cos(-t)\sin(-t)$$

$$\therefore \cos(-\theta) = \cos \theta ; \sin(-\theta) = -\sin \theta$$

$$x(-t) = \cos t - \sin t - \cos t \sin t$$

$$x_e(t) = \frac{1}{2} \{ \cancel{\cos t} + \sin t + \cancel{\cos t} \cancel{\sin t} + \cancel{\cos t} - \cancel{\sin t} - \cancel{\cos t} \cancel{\sin t} \}$$

$$= \frac{1}{2} (2 \cos t)$$

$$x_e(t) = \cos t$$

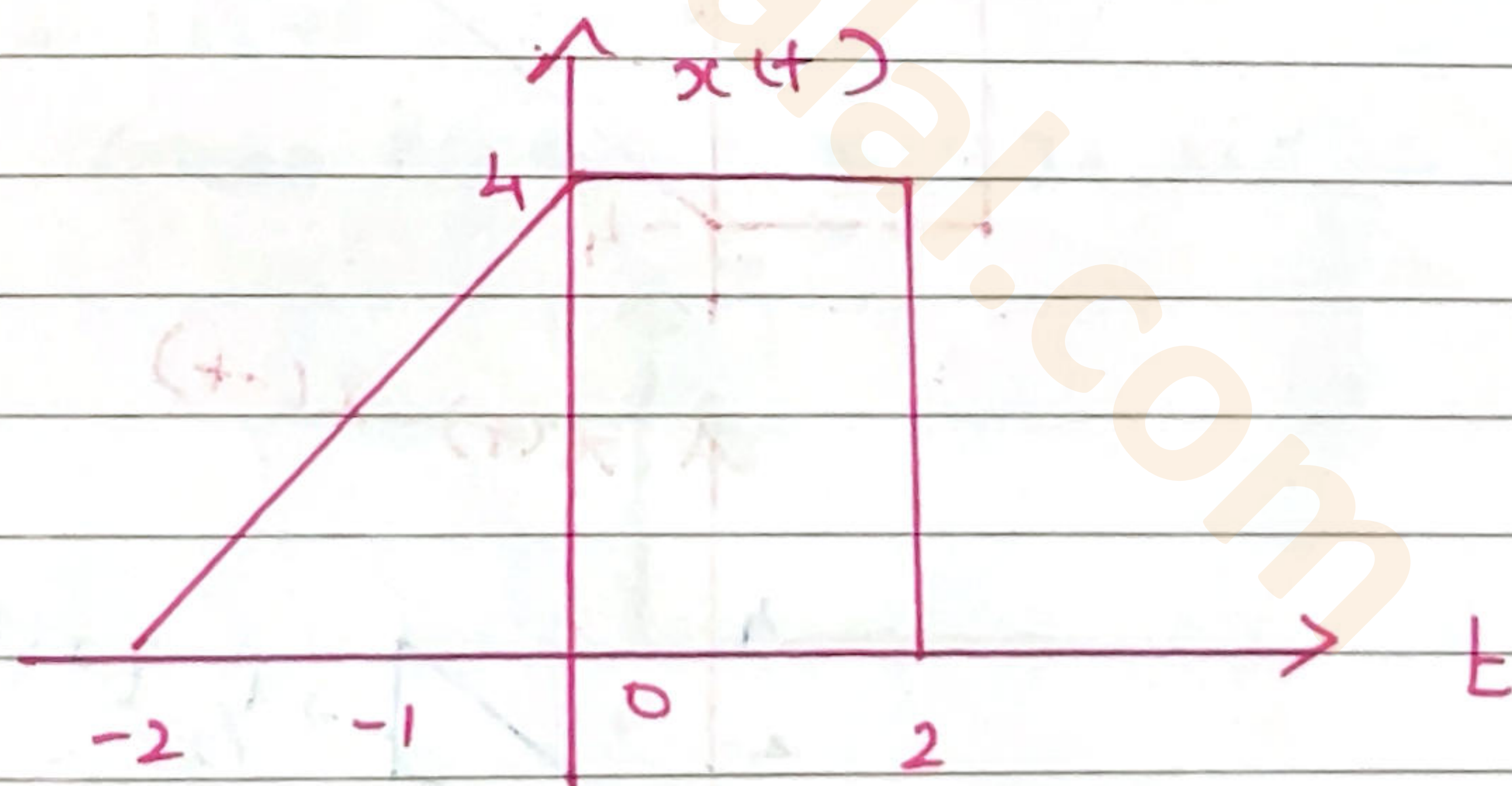
$$x_o(t) = \frac{1}{2} \{ \cancel{\cos t} + \sin t + \cancel{\cos t} \sin t - \cancel{\cos t} + \sin t + \sin t \cancel{\cos t} \}$$

$$= \frac{1}{2} \{ 2 \sin t + 2 \cos t \sin t \}$$

$$= \frac{2}{2} (\sin t + \cos t \sin t)$$

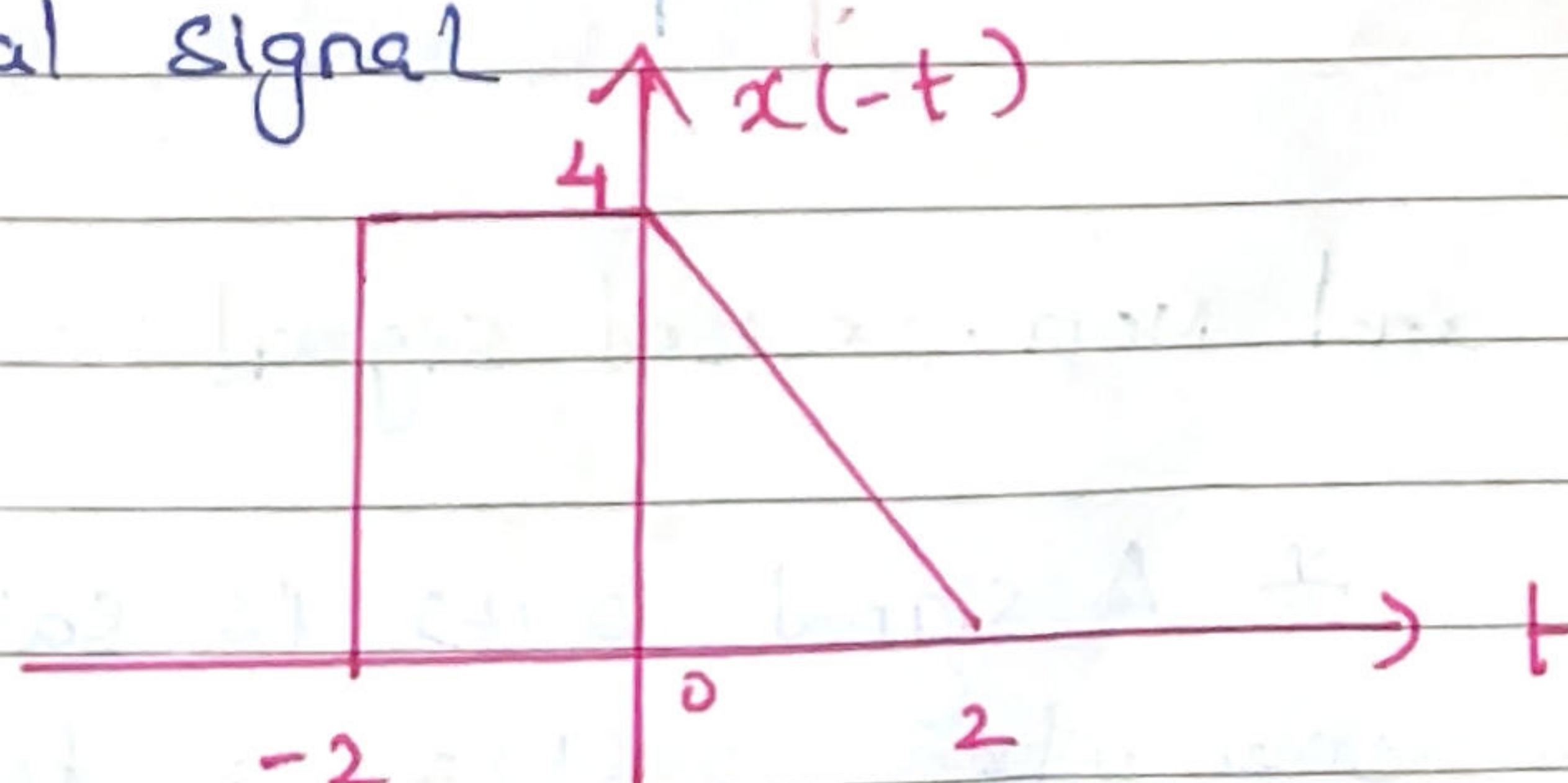
$$x_o(t) = \sin t + \sin t \cos t$$

2. Find even and odd components:-

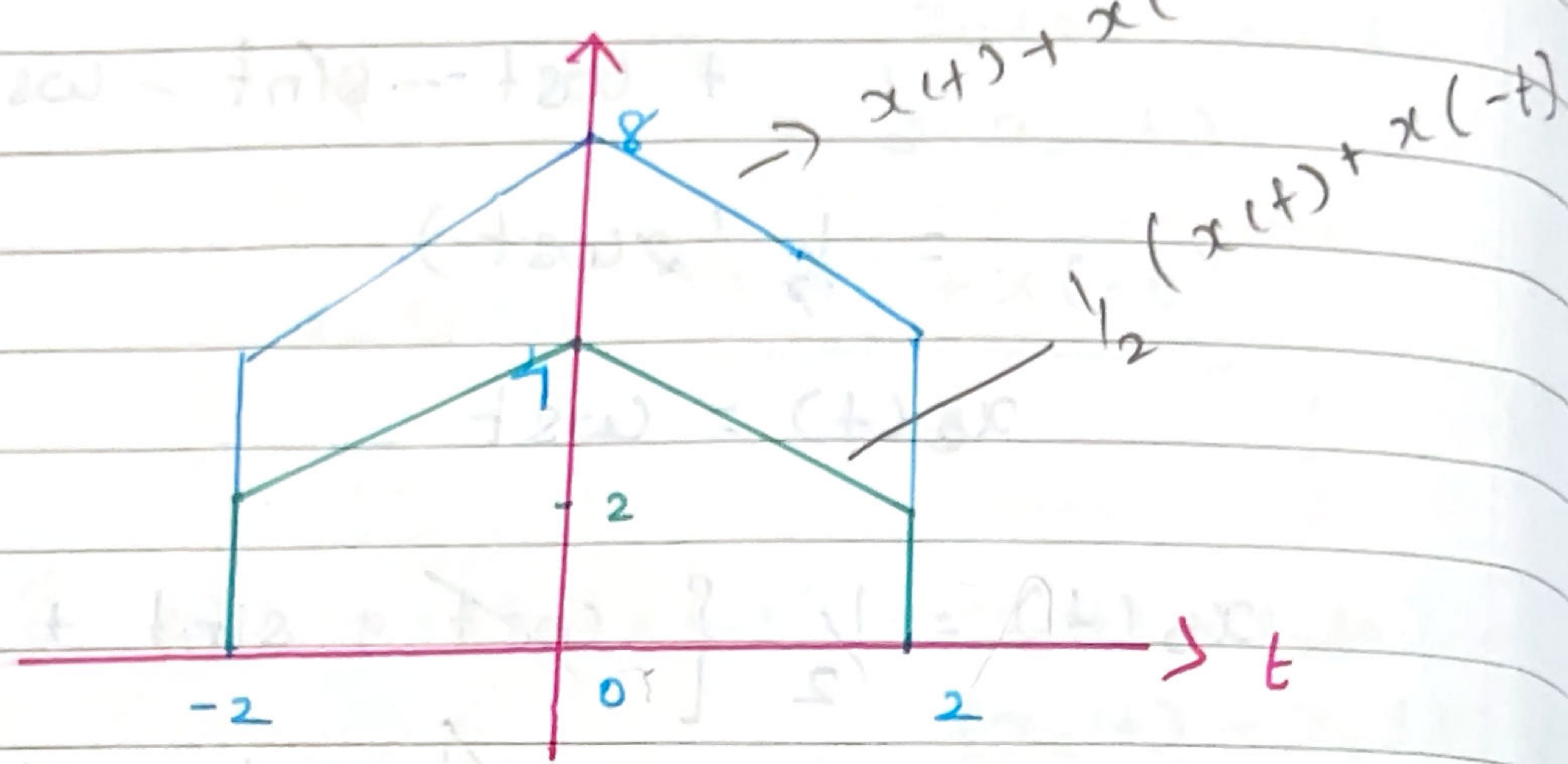


Solution:-

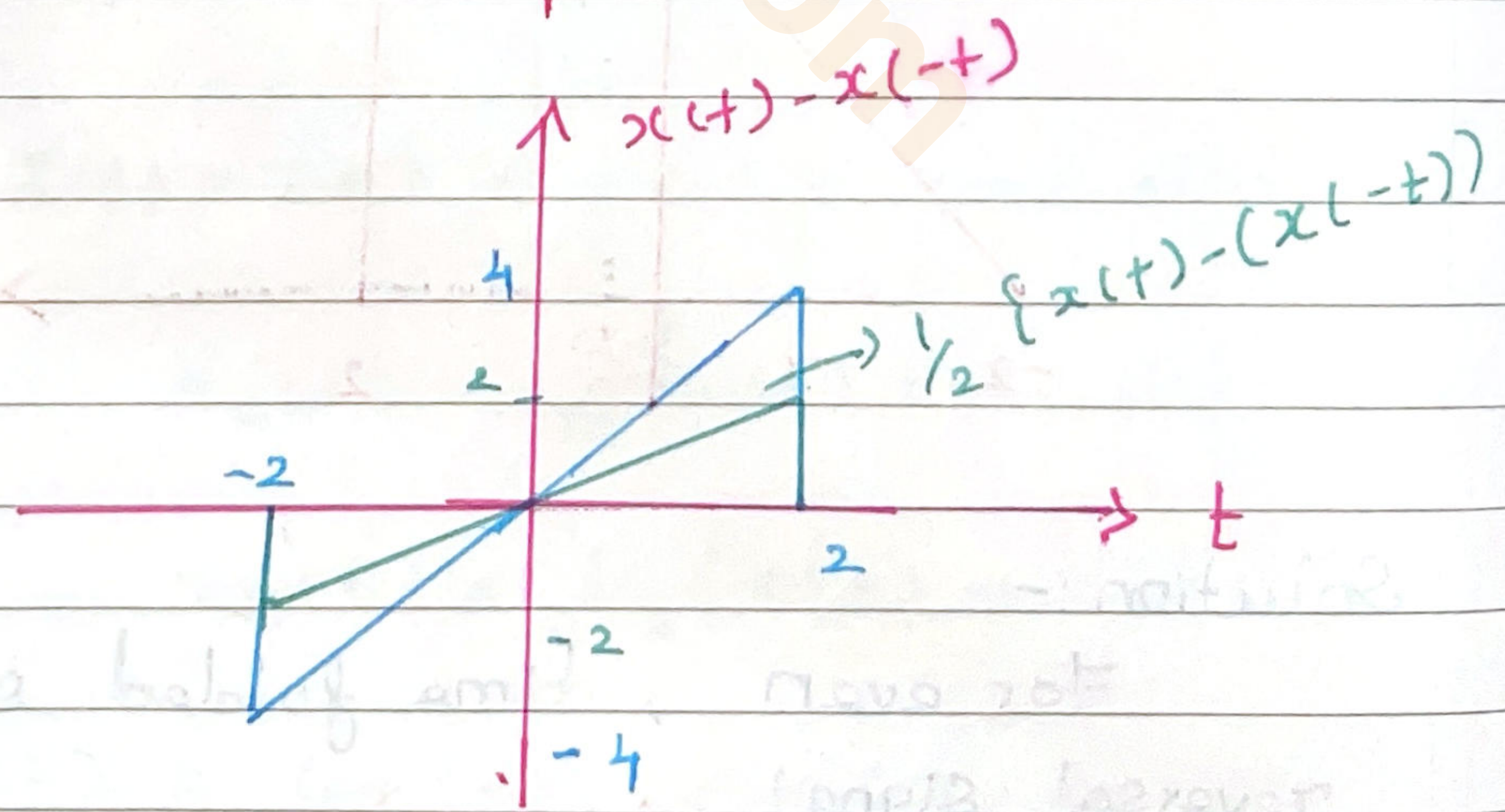
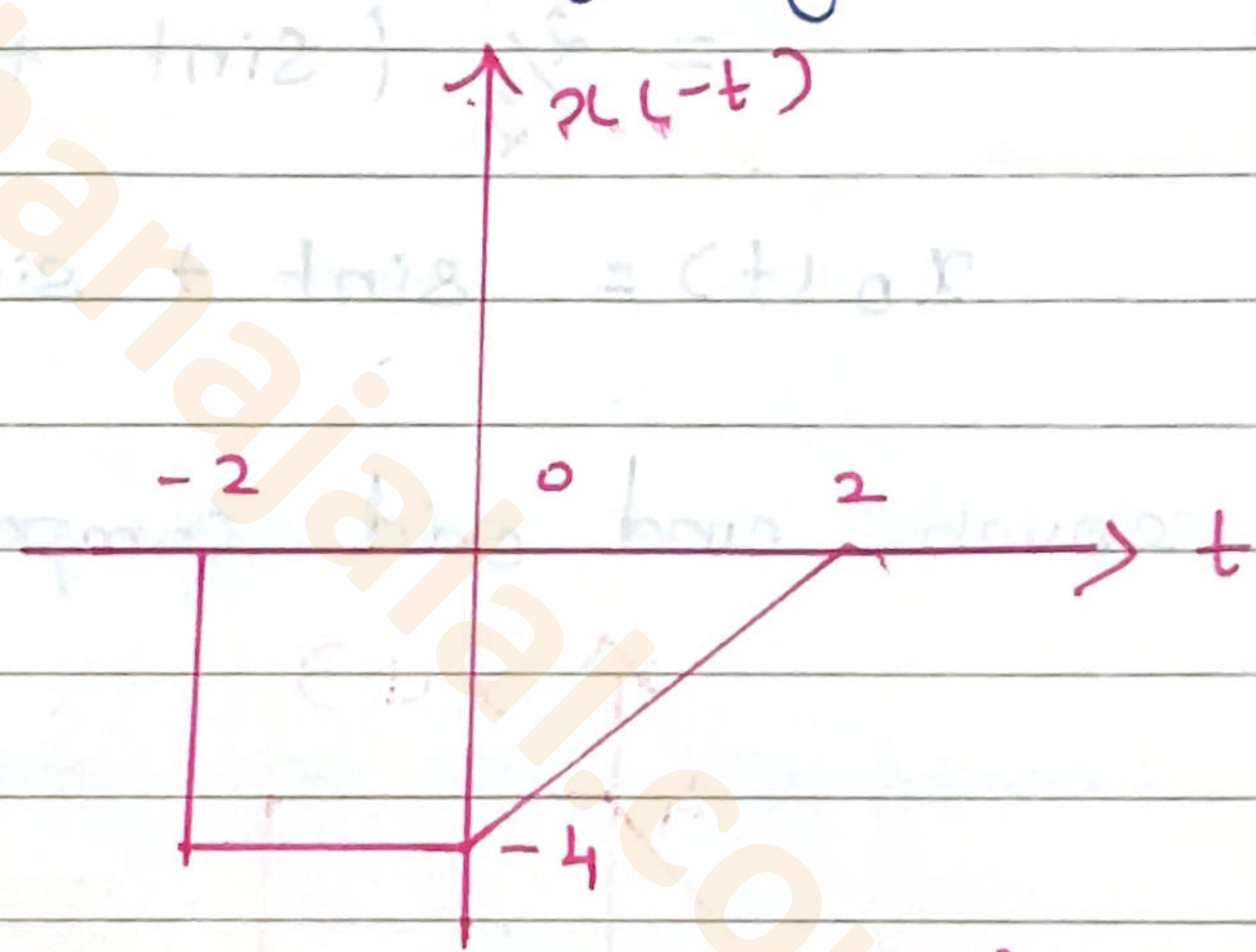
For even, time folded signal (or) reversal signal



For $x(t) + x(-t)$



For odd $\rightarrow$ amplitude folding of the time folding signal.



Causal and Non-causal signal :-

* A signal $x(t)$ is said to be casual only when $x(t) = 0$; for $t < 0$, that means casual signals are right sided

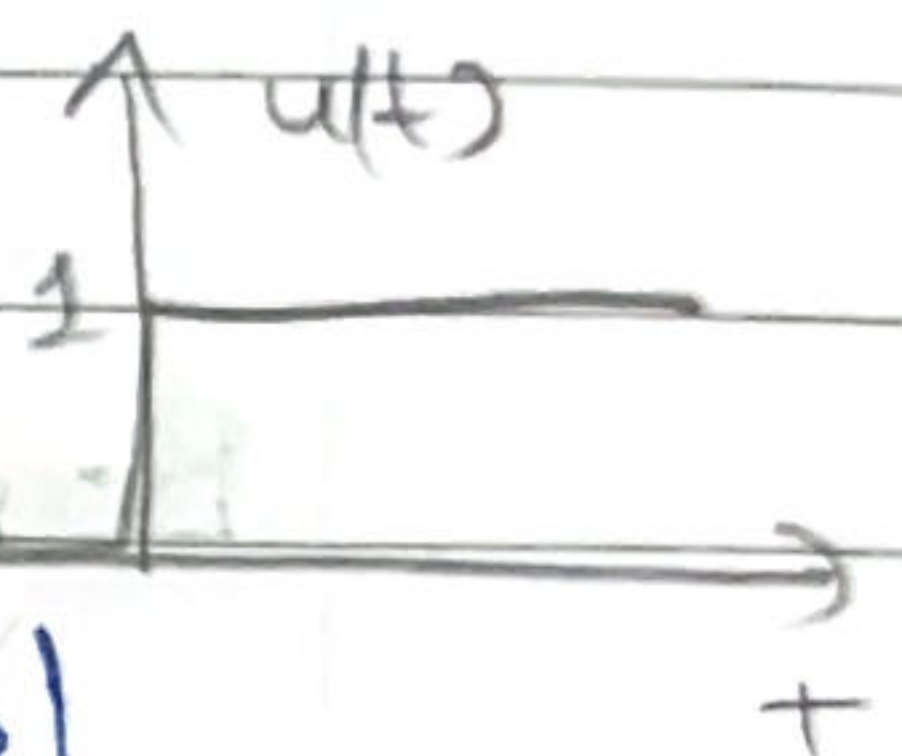
* A signal $x(t)$ is said to be anti-causal only when $x(t) = 0$ for $t > 0$, that means anti-causal signals are left-sided.

* A signal $x(t)$ is said to be non-causal only when $x(t) \neq 0$ for $t < 0$ i.e., non-causal signals may be both left-sided. All anti-causal signals come under non-causal signals.

Eg:-

a) $x_1(t) = u(t)$

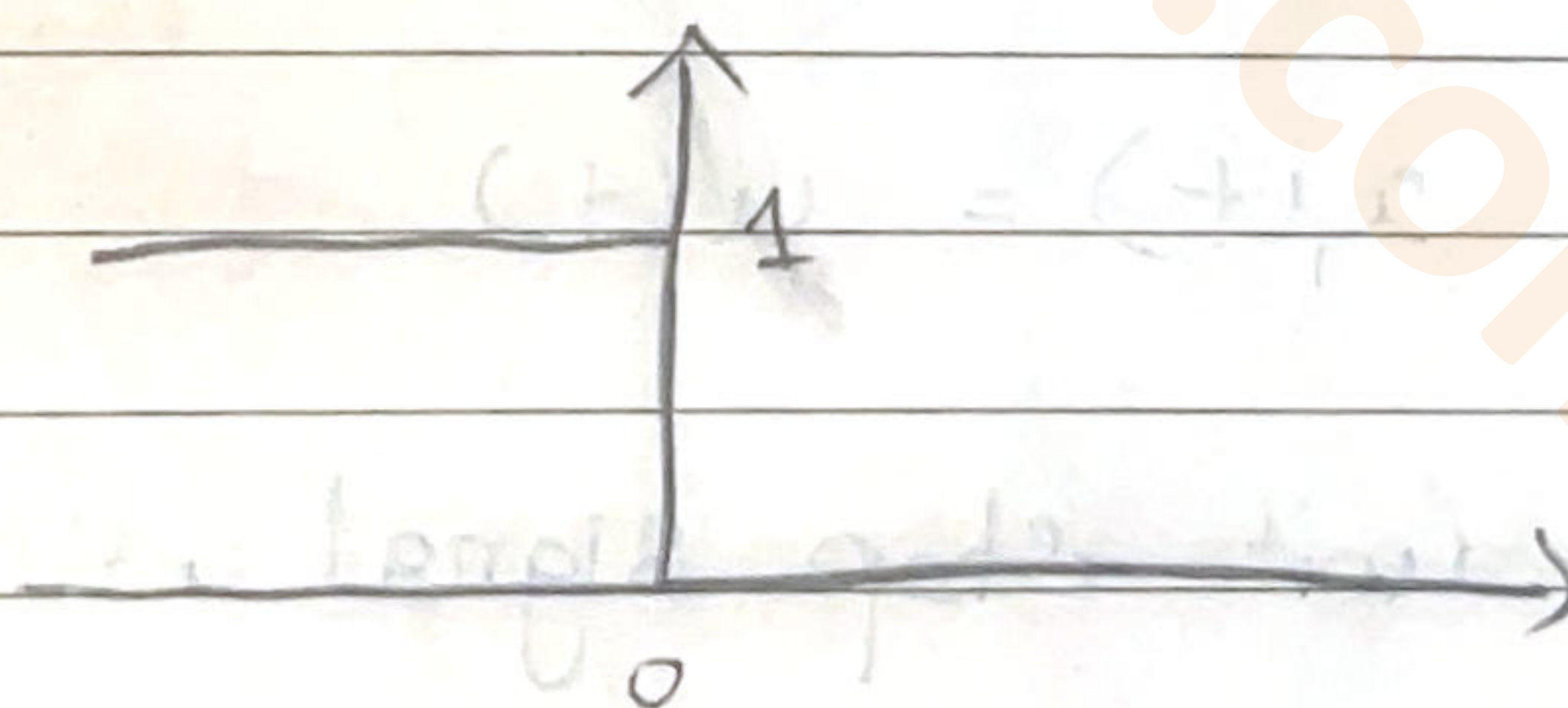
Given signal is a unit step signal



$\therefore x_1(t) = \text{causal signal because it is right-sided.}$

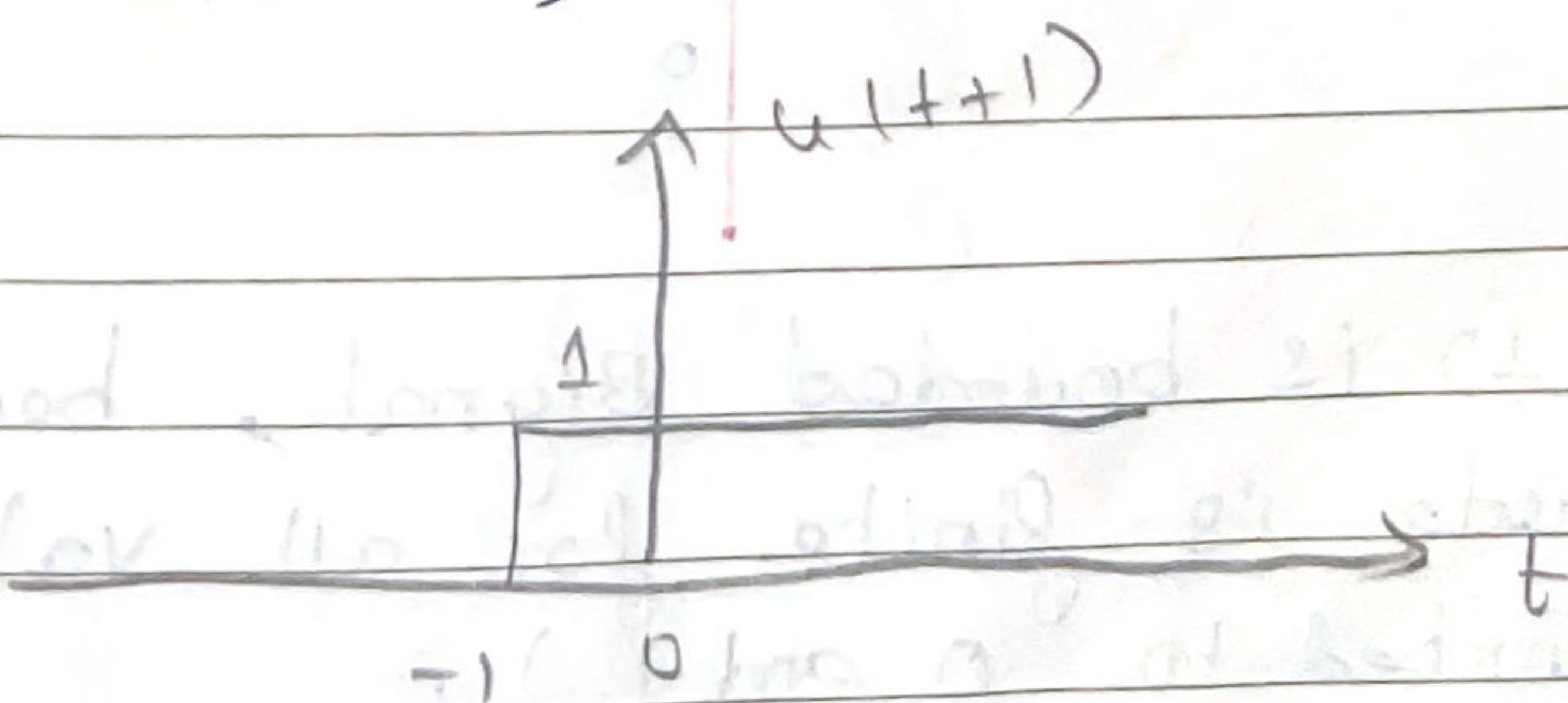
b) $x_2(t) = u(-t)$

Given signal $x_2(t) = u(-t)$



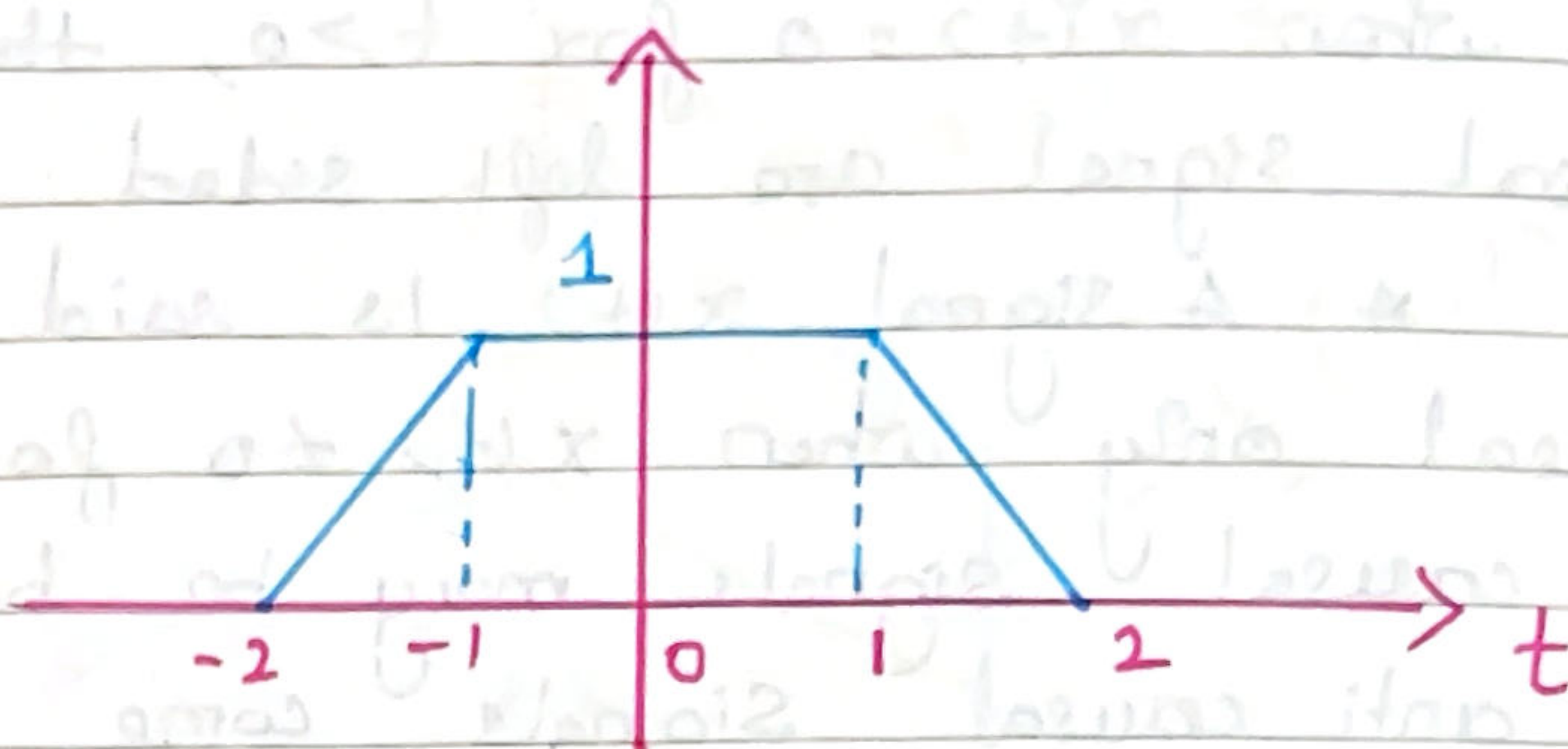
$\therefore x_2(t)$ is anti-causal (or) non-causal signal because it is left-sided.

c) $x_3(t) = u(t+1)$



$x_3(t)$ is non-causal signal, because it is both-sided.

d) $x_4(t) = r(t+2) - r(t+1) - [r(t-1) - r(t-2)]$



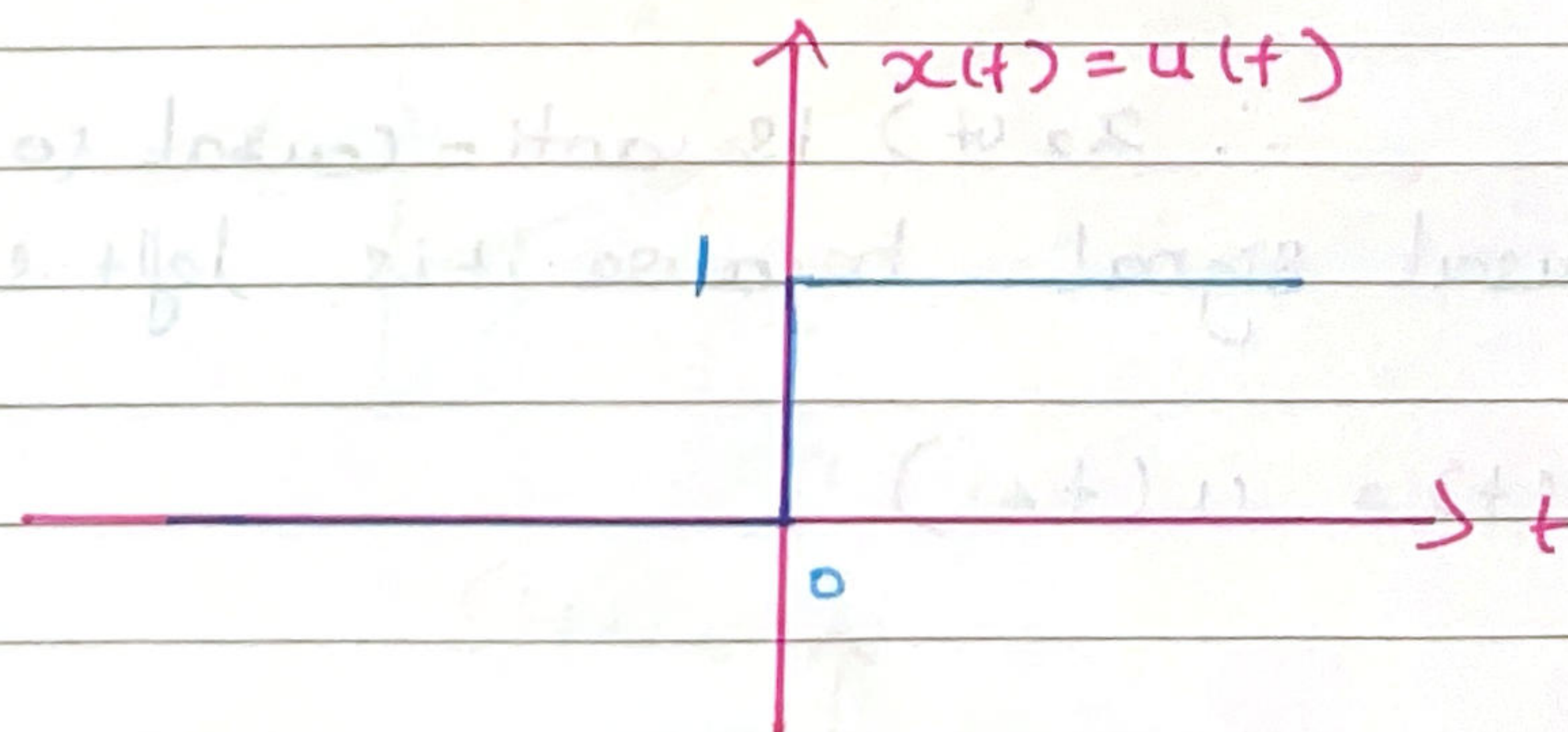
$x_4(t)$ is non-causal signal, because it is both sided.

Bounded and unbounded signals:-

A signal $x(t)$ is said to be bounded only when the amplitude of $x(t)$ is finite for all values of 't' over the range $-\infty \leq t \leq \infty$. Condition for a bounded signal is $|x(t)| < \infty$, $-\infty \leq t \leq \infty$.

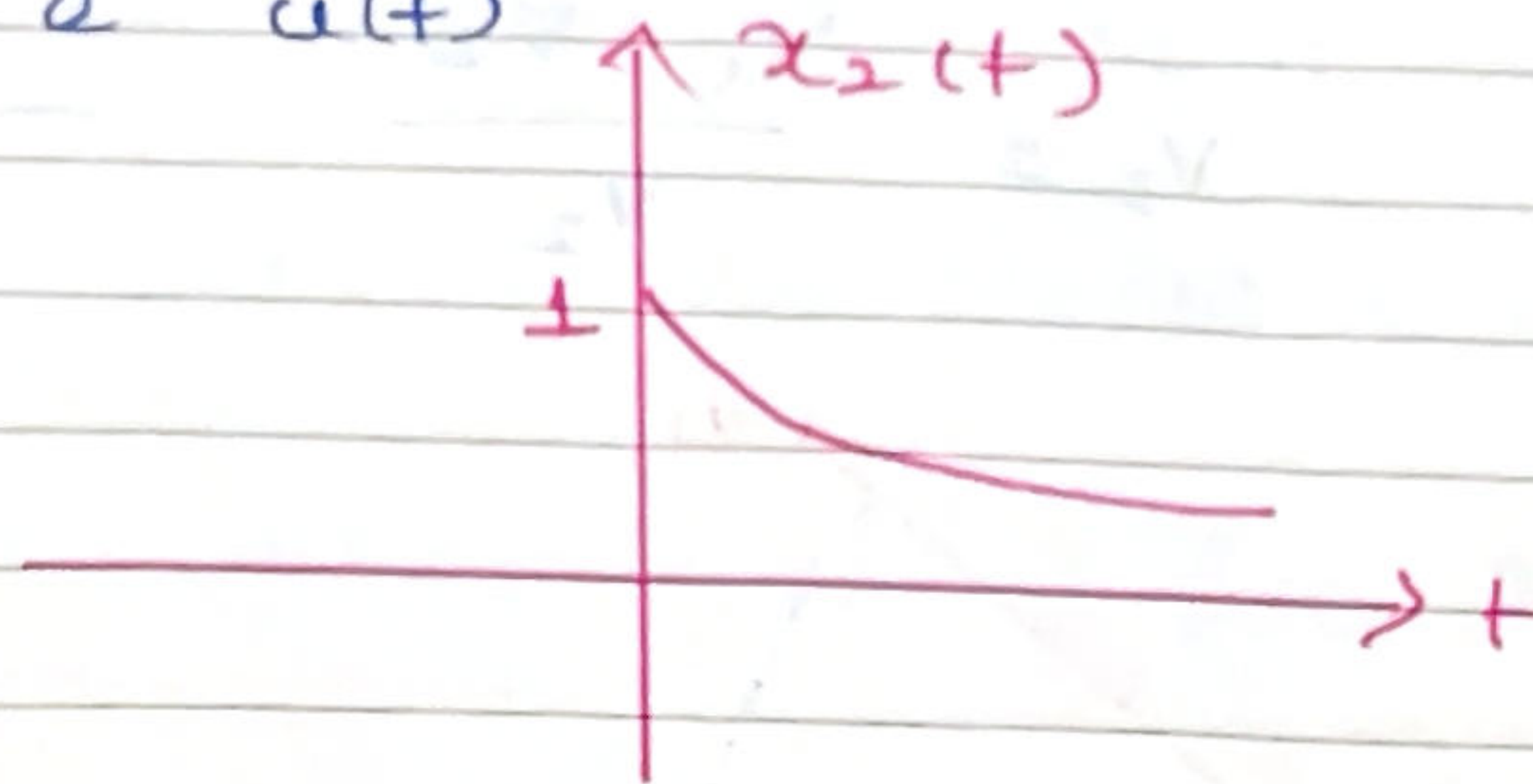
Eg:- a) $x_1(t) = u(t)$

Given: unit step signal,



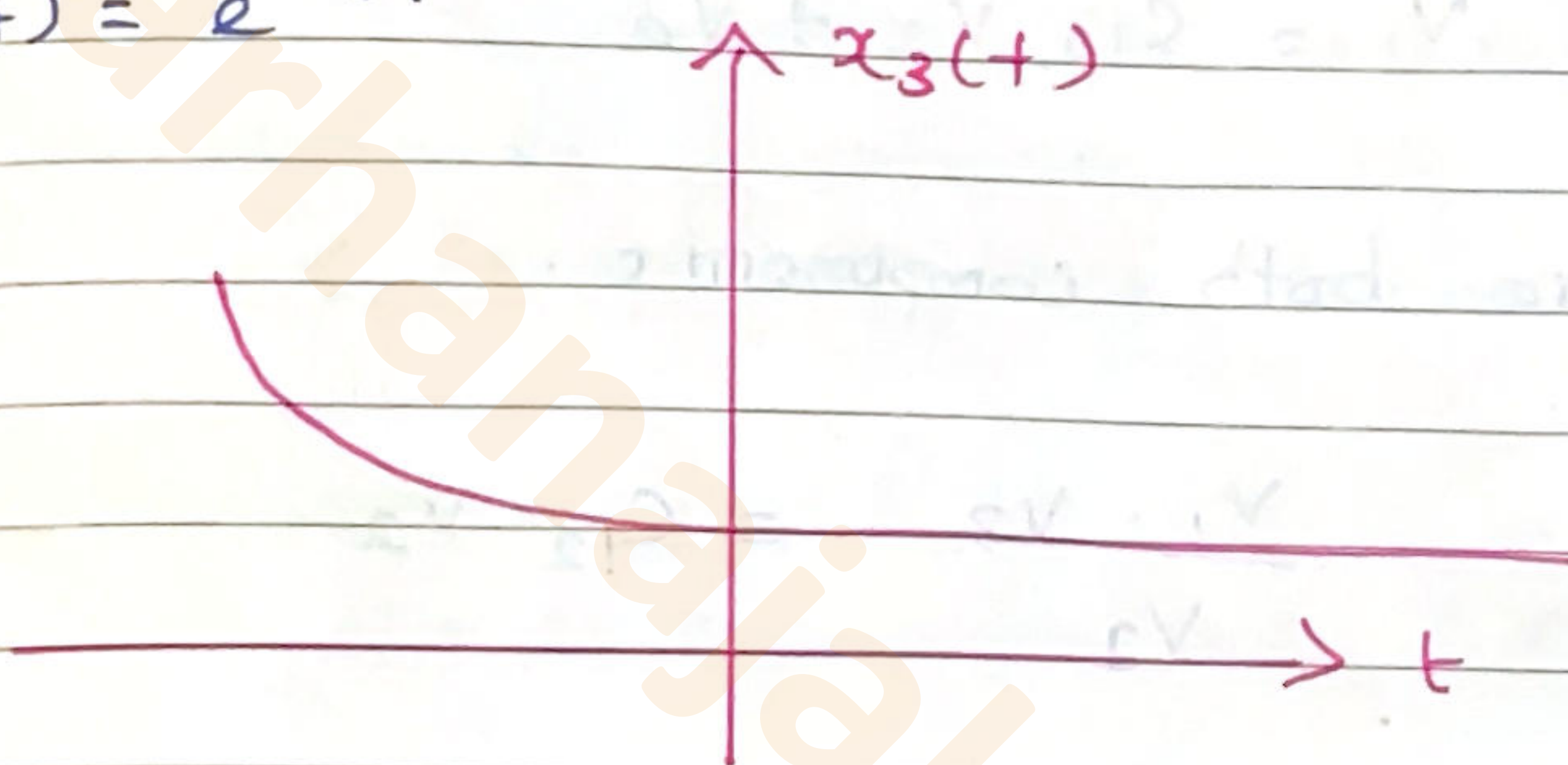
$x_1(t)$ is bounded signal, because its amplitude is finite for all values of t (limited to 0 and 1).

b) $x_2(t) = e^{-2t} u(t)$



$x_2(t)$ is bounded signal, because its amplitude is finite for all values of t (limited to 0 and 1)

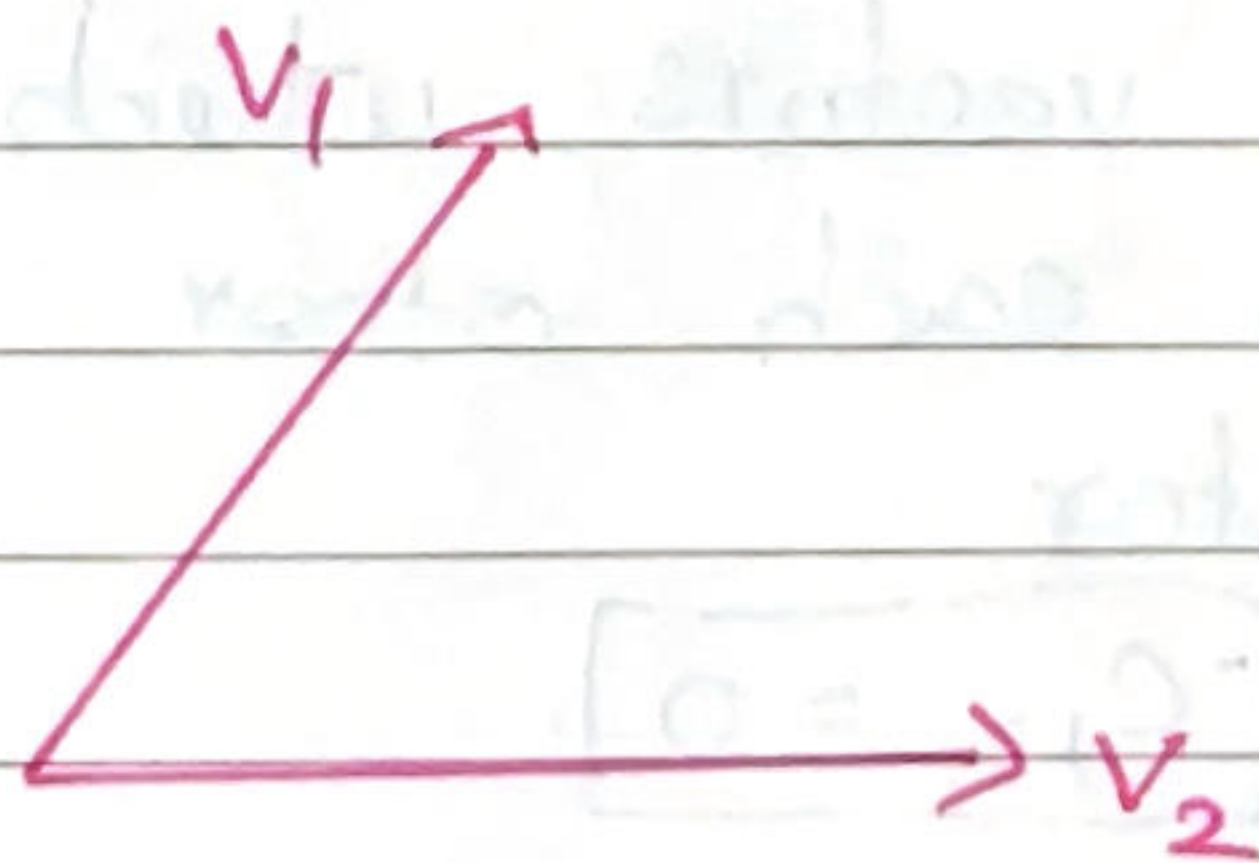
c) $x_3(t) = e^{-2t}$



$x_3(t)$ is unbounded signal, because its amplitude is infinity at $t = -\infty$.

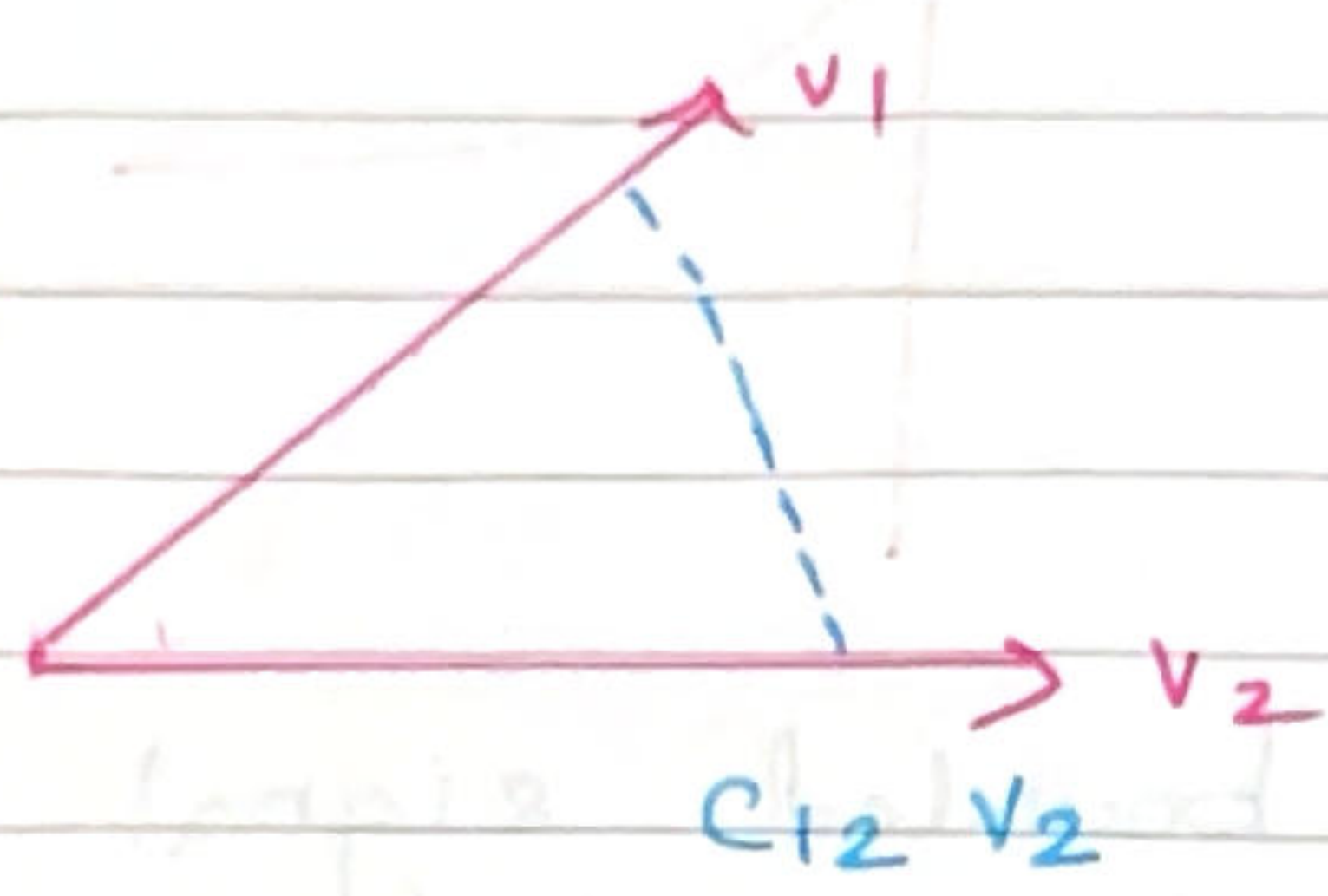
Analogy b/w vectors and signals :-

Vectors can be defined as a quantity, which has both magnitude & direction.



The component of vector v_1 along v_2 is given by

$$V_2 = \frac{V_1 \cdot V_2}{V_2} \quad \text{--- (1)}$$



$$V_1 \cong c_{12} V_2 \quad \text{--- (2)}$$

$$V_1 = c_{12} V_2 + V_e$$

Compare both components.

$$\frac{V_1 \cdot V_2}{V_2} = c_{12} \cdot V_2$$

$$c_{12} = \frac{V_1 \cdot V_2}{V_2^2}$$

When two vectors perpendicular to each other (or) dot product is 0

$$c_{12} = 0$$

Signals :-

* The vectors which are mutually perpendicular to each other they are called Orthogonal vector.

$$c_{12} = 0$$

lets take two signal $x_1(t)$ and $x_2(t)$ with limits $t_1 < t < t_2$

$$x_1(t) \approx c_{12} x_2(t)$$

$$x_1(t) = c_{12} x_2(t) + x_e(t)$$

Error signal :-

$$x_e(t) = x_1(t) - c_{12} x_2(t)$$

Select c_{12} such that $x_e(t)$ as minimum as possible.

The error can be Negative (or) positive one.

Generally to minimise the error function we consider a function.

* Mean Square error.

MEAN Square error :-

It is represented as $(\mathcal{E})$

In every approximation process, there will be some error and error due to the approximation can be computed through Mean square error (MSE). It is denoted as $(\mathcal{E})$ and it can be computed from the formula.

$$\mathcal{E} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [x_e(t)]^2 dt$$

Minimisation of MSE :-

$$\text{w.k.r } x_e(t) = x_1(t) - c_{12} x_2(t)$$

$$\begin{aligned} \mathcal{E} &= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [x_1(t) - c_{12} x_2(t)]^2 dt \\ &= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [x_1^2(t) + c_{12}^2 x_2^2(t) - 2c_{12} x_1(t) x_2(t)] dt \end{aligned}$$

To minimize the mean square error. Compute the constant by differentiating $\mathcal{E}$ with respect to c_{12} is equal to zero

$$\text{i.e., } \frac{d\mathcal{E}}{dc_{12}} = 0$$

$$\frac{d\mathcal{E}}{dc_{12}} = \frac{d}{dc_{12}} \left[\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x_1^2(t) + c_{12}^2 x_2^2(t) - 2c_{12} x_2(t) x_1(t) dt \right]$$

Rearranging the terms.

$$\begin{aligned} &= \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} \frac{d}{dc_{12}} x_1^2(t) dt + \int_{t_1}^{t_2} \frac{d}{dc_{12}} c_{12}^2 x_2^2(t) dt - \int_{t_1}^{t_2} \frac{d}{dc_{12}} 2c_{12} x_1(t) x_2(t) dt \right] \end{aligned}$$

$$\begin{aligned} &= \frac{1}{t_2 - t_1} \left[0 + \int_{t_1}^{t_2} 2c_{12} x_2^2(t) dt - 2 \int_{t_1}^{t_2} x_1(t) x_2(t) dt \right] \end{aligned}$$

now $\frac{d\epsilon}{dC_{12}} = 0$

$$\frac{1}{t_2 - t_1} \left[2 \int_{t_1}^{t_2} C_{12} x_2^2(t) dt - 2 \int_{t_1}^{t_2} x_1(t) x_2(t) dt \right] = 0$$

$$2 \int_{t_1}^{t_2} C_{12} x_2^2(t) dt = 2 \int_{t_1}^{t_2} x_1(t) x_2(t) dt$$

$$C_{12} \int_{t_1}^{t_2} x_2^2(t) dt = \int_{t_1}^{t_2} x_1(t) x_2(t) dt$$

$$C_{12} = \frac{\int_{t_1}^{t_2} x_1(t) x_2(t) dt}{\int_{t_1}^{t_2} x_2^2(t) dt}$$

Condition for orthogonal signals:-

If the signals are orthogonal then the scalar (or) dot product is '0'

ie) $C_{12} = 0$

$$\Rightarrow \int_{t_1}^{t_2} x_1(t) x_2(t) dt = 0$$

We know that signals are analogous to vectors (ie) if two signals $x_1(t)$ & $x_2(t)$ are

orthogonal over the intervals (t_1, t_2) then

$$\int_{t_1}^{t_2} x_1(t) x_2(t) dt = 0$$

It is the condition for orthogonal signal.

NOTE:-

* All sinusoidal signals, $\sin(\omega t)$ and $\cos(\omega t)$ are orthogonal signals.

* All complex exponential signals, $x(t) = Ae^{-j\omega t}$ & $y(t) = Ae^{j\omega t}$ are orthogonal signals.

Eg:-

① Show that the following signals are orthogonal over an interval $[0, 1)$, $x_1(t) = 2$, $x_2(t) = \sqrt{3}(1-2t)$.

Solution:-

W.K.T

$$\int_{t_1}^{t_2} x_1(t) x_2(t) dt = 0$$

$$\int_0^1 \{2 \cdot \sqrt{3}(1-2t)\} dt = 0$$

$$\int_0^1 (2\sqrt{3} - 4\sqrt{3}t) dt = 0$$

$$= \int_0^1 2\sqrt{3} dt - \int_0^1 4\sqrt{3} t dt$$

$$\int x \cdot dx = \frac{x^2}{2}$$

$$= [2\sqrt{3}t]_0^1 - [4\sqrt{3} \frac{t^2}{2}]_0^1$$

$$= 2\sqrt{3} [1-0] - 4\sqrt{3} [\frac{1}{2} - 0]$$

$$= 2\sqrt{3} - 2\sqrt{3}$$

$$= 0$$

Hence the given functions are orthogonal over a interval $[0, 1]$

Orthogonal Signal space:-

Mutually orthogonal function: also called as mutually orthogonal set :-

Consider a set of 'n' function

$\{x_1(t), x_2(t), \dots, x_n(t)\}$ which are orthogonal to one another over (t_1, t_2)

The condition for orthogonality is complex function

$$\int_{t_1}^{t_2} x_1^*(t) x_2(t) dt = 0 \text{ (or)}$$

$$\int_{t_1}^{t_2} x_1(t) x_2^*(t) dt = 0$$

if $\{x_1(t), x_2(t), \dots, x_n(t)\}$

closed, complex set of orthogonal functions

$$\int_{t_1}^{t_2} x_m^*(t) x_n(t) dt = 0 \quad (or)$$

$$\int_{t_1}^{t_2} x_m(t) x_n^*(t) dt = 0$$

} $m \neq n$

if $m = n$

$$\int_{t_1}^{t_2} x_m^*(t) x_m(t) dt$$

$$E_m = \int_{t_1}^{t_2} |x_m(t)|^2 dt \quad \text{--- Energy formula}$$

↓
 $x_m(t)$

Problems :-

- (1) If $x(t)$ and $y(t)$ are orthogonal, then show that $E_{x+y} = E_x + E_y$ is energy of the signal $x(t) + y(t)$. E_x is the energy of signal $x(t)$ & E_y is the energy of the signal $y(t)$

Solution :-

Given :-

$$E_x = \int_{-\infty}^{\infty} x^2(t) dt$$

$$E_y = \int_{-\infty}^{\infty} y^2(t) dt$$

The energy signal $x(t) + y(t)$ is

$$E_{x+y} = \int_{-\infty}^{\infty} (x(t) + y(t))^2 dt$$

$$= \int_{-\infty}^{\infty} x^2(t) dt + \int_{-\infty}^{\infty} y^2(t) dt +$$

$$2 \int_{-\infty}^{\infty} x(t)y(t) dt$$

Since $x(t)$ and $y(t)$ are orthogonal

$$\text{w.k.t } \int_{-\infty}^{\infty} x(t)y(t) dt = 0$$

$$\therefore E_{x+y} = \int_{-\infty}^{\infty} x^2(t) dt + \int_{-\infty}^{\infty} y^2(t) dt$$

$$E_{x+y} = E_x + E_y$$

② Prove that the set of exponentials $\{1, e^{\pm j\omega_0 t}, e^{\pm j2\omega_0 t}, \dots\}$ is orthogonal

Over any interval To

Solution :-

Given:-

The set of exponentials are complex signals.

$$w.k.t \int_{t_1}^{t_2} x_m(t) x_n^*(t) dt.$$

$$\text{Let } x_m(t) = e^{j m \omega_0 t} \text{ \& }$$

$$x_n(t) = e^{j n \omega_0 t}$$

for any interval T_0

$$\int_t^{t+T_0} e^{j m \omega_0 t} (e^{j n \omega_0 t})^* dt$$

$$= \int_t^{t+T_0} e^{j(m-n)\omega_0 t} dt$$

$$= \frac{1}{j(m-n)\omega_0} \left[e^{j(m-n)\omega_0 t} \right]_t^{t+T_0}$$

$$= \frac{1}{j(m-n)\omega_0} \left[e^{j(m-n)\omega_0(t+T_0)} - e^{j(m-n)\omega_0 t} \right]$$

$$= \frac{1}{j(m-n)\omega_0} e^{j(m-n)\omega_0 t} \left[e^{j(m-n)\omega_0 T_0} - 1 \right]$$

Since $\omega_0 T_0 = 2\pi$

$$e^{j(m-n)2\pi} = 1 \text{ for any } m \& n$$

Hence :

$$\int_t^{t+T_0} e^{j m \omega_0 t} (e^{j n \omega_0 t})^* dt = 0$$

if $m = n$, then

$$\int_t^{t+T_0} e^{j m \omega_0 t} (e^{j n \omega_0 t})^* dt =$$

$$\int_t^{t+T_0} dt = T_0$$

$$\therefore \int_t^{t+T_0} e^{j m \omega_0 t} (e^{j n \omega_0 t})^* dt = 0 \text{ for } m \neq n$$

$$= T_0 \text{ for } m = n.$$

$\therefore$ The set of exponentials is orthogonal.

Parseval's relation:-

The total energy (or) power of a signal in the time domain is equal to its total energy (or) power in the frequency domain.

For

Continuous-time Fourier Transform.

For continuous-time a periodic signal $x(t)$ with a Fourier transform $X(\omega)$, the energy conservation is represented as,

$$\int_{-\infty}^{\infty} (x(t))^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

If ~~two~~ dealing with two different signals

$x_1(t)$ and $x_2(t)$ the Parseval's relation is

$$\int_{-\infty}^{\infty} x_1(t) x_2^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(j\omega) X_2^*(j\omega) d\omega$$

Discrete time signals :-

In Discrete time Fourier transform where $X(\omega)$ is a continuous function of frequency with a period of 2π , the relation is:

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(j\omega)|^2 d\omega$$

For finite length N (using DFT) it is translated to

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

Parseval's relation

$$\text{If } F[x(t)] = X(j\omega)$$

then

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

Proof :-

$$|x(t)|^2 = x(t) x^*(t)$$

Integrating both sides over $-\infty$ to ∞

W.K.T Energy

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t) \cdot x^*(t) dt \quad \text{--- (1)}$$

$$x(t) = F^{-1}[X(j\omega)] = \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$x^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) e^{-j\omega t} d\omega$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) e^{-j\omega t} d\omega \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) X(j\omega) d\omega$$

$$\therefore \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

Hence proved.